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Some New Directions in Fixed Point Theory

by

Usman Shehzad

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Dedicated to my (Late) Father, my Loving Mother, my Wife and my Beloved Daughter Abbrish Shehzad.

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CERTIFICATE OF APPROVAL

This is to certify that the research work presented in the dissertation, entitled "**Some New Directions in Fixed Point Theory**" was conducted under the supervision of **Dr. Samina Batul**. No part of this dissertation has been submitted anywhere else for any other degree. This dissertation is submitted to the **Department of Mathematics, Capital University of Science and Technology** in partial fulfillment of the requirements for the degree of Doctor in Philosophy in the field of **Mathematics**. The open defence of the dissertation was conducted on **January 20, 2026**.

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I, **Usman Shehzad** hereby state that my PhD dissertation titled "**Some New Directions in Fixed Point Theory**" is my own work and has not been submitted previously by me for taking any degree from Capital University of Science and Technology, Islamabad or anywhere else in the country/abroad.

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Abstract

This dissertation introduces and explores several new fixed point results for fuzzy and multivalued mappings within various generalized metric structures. A key contribution is the formulation of the (P, ψ)-type almost contraction condition, specifically designed for fuzzy mappings in complete b-metric spaces. The resulting fixed point theorems are supported by illustrative examples and are applied to demonstrate the existence of solutions to second-order nonlinear boundary value problems. This study investigates fixed point results for fuzzy mappings under generalized contraction conditions within double controlled metric spaces, i.e., Θ-fuzzy double controlled contractions and Θ-fuzzy almost generalized double controlled contraction mappings. These generalized mappings ensure the existence and uniqueness of fixed points under appropriate assumptions and significantly expand classical results. Detailed examples are provided to validate the theoretical findings.

Additionally, we present fuzzy fixed point results based on integral-type Θ-contractions in the setting of b-metric spaces. Their applicability is shown through their use in solving stochastic Volterra integral equations. Finally, common fixed point theorems are established for multivalued mappings using rational type Nashine-Wardowski-Feng-Liu contractions in orbitally complete controlled metric spaces.

Throughout the dissertation, numerous corollaries are derived to demonstrate that the main results unify and generalize a broad class of known fixed point theorems. The contributions of this work offer a comprehensive framework that advances fixed point theory and its applications in nonlinear analysis.

Contents

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Abbreviations

FM	Fuzzy mapping
FP	Fixed point
FS	Fuzzy set
$F_{\Theta}GAC$	Fuzzy Θ-type generalized almost contraction
LSC	Lower semi continuous
MS	Metric space
MM	Multivalued mapping
$\Theta - \mathtt{FDCCM}$	Θ-fuzzy double controlled contraction mapping
$\Theta - \mathtt{FAGDCCM}$	Θ-fuzzy almost generalized double controlled contraction mapping

Symbols

$\mathbb{C}$	Set of complex numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{N}$	Set of natural numbers
H	Hausdorff distance
$\mathfrak{CB}(\mathsf{G})$	Closed and bounded subsets of G
d	Distance between two points
$\mathsf{P}(\mathsf{G})$	Set of all closed and compact subsets of G

Chapter 1

Introduction

Mathematics occupies a central role in scientific inquiry and is frequently considered the cornerstone of all scientific disciplines due to its fundamental contribution to modeling and interpreting natural phenomena. This discipline is further divided into numerous branches, each possessing its own unique significance and applications. Functional analysis, a major branch of classical mathematical analysis which emerged in the late 19th century and solidified its theoretical framework during the 1920s and 1930s. Today, it stands as one of the most profound and impactful areas of mathematical research, with extensive applications in both pure and applied sciences. Functional analysis primarily investigates functionals, or mappings defined on infinite-dimensional spaces, and its theoretical reach has profoundly influenced diverse mathematical domains. Notably, its principles are widely employed in numerical analysis, particularly in error estimation for polynomial interpolation and finite difference methods (see, e.g.,[1–3]).

Among the many important developments in functional analysis, fixed point(FP) theory stands out as a significant and fruitful outcome. For several decades, it has been regarded as a dynamic and evolving area within nonlinear functional analysis. FP theory is inherently interdisciplinary, drawing upon elements from geometry, topology, and analysis, and it continues to be a rich field for ongoing research and innovation.

In numerous domains of mathematics and applied sciences, determining the existence of solutions to nonlinear problems constitutes a fundamental research challenge. FP theory offers a robust methodological framework for addressing this question by guaranteeing the existence of FPs, which often correspond to solutions of such nonlinear equations. The foundational ideas of this theory date back to 1866, when Poincaré and Birkhoff [4] in 1886, introduced a FP theorem asserting that any area-preserving, orientation-preserving homeomorphism of an annulus, which rotates the two boundaries in opposite directions, must have at least two FPs. Subsequently, Brouwer [5] examined the equation $\Re_1(\zeta) = \zeta$ and, in 1910, resolved it by proving a FP result. Brouwer demonstrated that any continuous mapping from an n-simplex to itself possesses a FP. This theorem extends to the unit ball $\mathbb{B}^n$ or any compact convex subset of $\mathbb{R}^n$, which can be used interchangeably with the n-simplex. Later, Kakutani [6] expanded on these ideas, contributing to the theories for further development.

FP theory is particularly notable for its use in successive approximation methods, offering three fundamental contributions:

(i): Establishing the existence of solutions for nonlinear problems.

(ii): Demonstrating the uniqueness of those solutions under certain conditions.

(iii): Designing iterative processes that converge to a FP, which represents the solution to the problem.

Foundational ideas in this area are attributed to several prominent mathematicians, including Liouville, Lipschitz, Cauchy, Peano, Fredholm, and most notably, Picard, whose contributions laid the groundwork for the field.

1.1 Metric FP Theory

The concept of a metric space(MS), which serves as a cornerstone in mathematical analysis, was first formulated by M. Fréchet in 1906 [7], who is credited as the pioneer of this concept. He developed the concept in an axiomatic way as a broad formulation of the classical Euclidean distance.

The metric concept plays a key role in various branches of mathematics, particularly in real, complex, and functional analysis. Due to its fundamental importance, the metric has been generalized and modified in many ways across different mathematical disciplines. FP theory has developed into a rich and dynamic field of mathematics, with significant contributions across both pure and applied branches. Rudin [8] emphasized completeness and continuity in analysis, elements essential for understanding FP results. Agarwal [9] extended FP techniques to differential equations and boundary value problems, demonstrating the practical impact of the theory. Kreyszig [10] integrated FP theorems into functional analysis, particularly in the context of Banach and Hilbert spaces, making them useful tools in operator theory. Knopp [11] contributed to understanding sequences and series, which underpin iterative processes used in FP applications.

The work of Morse and Cairns [12] incorporated topological insights that influence FP results in continuous mappings. Heinonen [13] explored potential theory in metric measure spaces, contributing to the nonlinear analysis framework that supports generalizations of FP results. Topological algebra, as studied by Dikranjan and Tholen [14], also intersects with FP theory when considering structures equipped with algebraic operations.

Bianchini [15] applied FP theory to the study of integral equations, especially those of Fredholm and Volterra types. Altogether, these works have established FP theory as an essential mathematical framework for demonstrating the existence and uniqueness of solutions in a wide range of problems, including boundary value problems, eigenvalue estimates, and stability analysis. The continued exploration of this field reflects its enduring importance in contemporary mathematical research. In its most abstract form, the FP approach often starts with a set of elements that fulfill specific axioms. Banach [16] played a crucial role by formalizing these ideas into a more generalized and abstract framework. His formulation expanded the applicability of FP theory beyond basic differential and integral equations, making it more suitable for a wide range of mathematical problems.

This principle states that every self-mapping $\Re_1$ on a complete MS (G, d), satisfying:

$$\mathsf{d}(\Re_1\zeta_1, \Re_1\zeta_2) \leq \lambda \mathsf{d}(\zeta_1, \zeta_2), \ \ \forall \ \ \zeta_1, \zeta_2 \in \mathsf{G} \text{ and } \lambda \in [0, 1),$$

has a unique FP. Banach introduced an iterative method aimed at locating the FP of a mapping, which encouraged researchers to adopt this strategy for proving the existence of solutions to differential and integral equations. The Picard iteration scheme, known for its clarity and effectiveness, is frequently used within the framework of Banach's contraction principle and is widely regarded as a foundational tool in FP theory.

The Banach contraction principle has had a profound impact on the development of metric FP theory. This result represented a significant advancement in solving nonlinear problems and became a valuable tool across various scientific and technical disciplines by offering a logical method for proving the existence of solutions.

Since its introduction, FP theory has expanded considerably. Its constructive approach has supported the creation of numerical techniques for computing FPs, contributing to its reputation as an active and important field of research. Banach's theorem gained prominence because of its wide ranging applications and adaptability. The ongoing refinement and extension of this principle are driven by its relevance in multiple branches of mathematics and science. After the initial formulation of Banach's principle, researchers began to explore its extensions in two approaches:

- (i): In the first approach, such results are obtained by modifying the structure of the underlying space.
- (ii): In the second approach, FP results are achieved by broadening the class of contraction conditions, leading to generalizations of the Banach FP theorem.

These efforts led to a wide range of generalizations, reflecting the principle's adaptability and foundational significance in various mathematical contexts. In the first approach Choudhury and Das [17] established a new contraction principle in Menger spaces. Similarly, one can find much literature where the underlying space is altered. For example, Oltra and Valero [18] used partially ordered spaces for generalizing certain FP results. In 1989, Bakhtin [19] presented the idea of b-MSs, offering a significant extension of the traditional MS framework. His approach involved modifying the classical triangle inequality. This marked what is arguably the earliest formal generalization of MSs. Among these generalizations, the b-MS has emerged as a particularly interesting and active area of research.

In recent decades, mathematicians have explored several structures built upon this idea, including rectangular MSs [20], rectangular b-MSs [21]. Later on Czerwik [22, 23] advanced this idea by employing a weaker version of the triangle inequality, thereby broadening the utility of b-MSs.

Another notable generalization, known as the extended b-MS, was proposed by Kamran et al. [24] in 2017. Several FP theorems based on the framework extended b-MS have since been established. Isik et al. [25] propose the structure of extended quasi b-metric-like spaces, which serves as a unifying generalization of both quasi metric-like spaces and quasi b-metric-like spaces. Du and Karapınar [26] extend the concept of a b-MS, by introducing the idea of a cone b-MS, and various FP theorems were subsequently developed within this generalized structure. Due [27] work is devoted to investigating the equivalence between the vectorial forms of FP theorems in generalized cone MSs and the scalar forms of such theorems in conventional MSs.

In particular, it is shown that the Banach contraction principles in standard MSs and in topological vector space-cone MSs are logically equivalent. Jleli and Samet [28] discuss the concept of G-MSs, which generalize standard MSs, along with existing FP results for contractive mappings defined in this framework.

Dosenovic et al. [29] analyzed multiplicative metrics and established FP results for various multiplicative contractions. Al Mezel et al. [30] demonstrate that FP results within the framework of complex-valued MSs can be derived as direct consequences of corresponding known results in the setting of associative MSs.

Specifically, it is shown that every complex-valued MS can be viewed as a particular case of a cone MS equipped with a normal cone. Dubey [31] and Kamaran [32] introduced generalizations in dislocated and Feng-type MSs respectively. Erhan et al. [33] investigated the existence and uniqueness of FPs for a broad class of $(\psi - \phi)$-contractive mappings within the framework of complete rectangular MSs. Huang and Zhang [34] formalized cone MSs, which are widely used in nonlinear analysis. Shehwar et al. [35] extended Caristi's result to C^* algebra valued MSs. Likewise, the notion of the Hausdorff distance was introduced by Felix Hausdorff [36] as an extension to the Banach contraction principle. Later on Malaiki et al. [37] introduced a new extension of b-MSs, called controlled MSs, by employing a control function on the right-hand side of the b-triangle inequality. Abdeljawad et al. [38] further expanded this framework by plugging two control function in the triangular inequality to introduce double controlled MS.

The second approach focuses on broadening the class of contraction conditions under which FP results hold. For example, Boyd and Wong [39] discussed non-linear contractions, by employing a function ψ defined on the closure of the range of MS. Arvanitakis [40] presented a proof of the generalized Banach contraction conjecture introducing the notion of J-continuity.

Ciric [41] introduced the concept of quasi-contractions, significantly extending the scope of Banach's contraction principle by relaxing its conditions. This result is further extended by Boriceanu [42] by proving the existence of a common FP satisfying ϕ contraction in b-MS for single and multi-valued mapping(MM)s. He also established FP result using contractive mappings with two b-metrics and multi-valued generalized contractions in b-MS. Caristi [43] proposed a characterization of weakly inward mappings is established based on a condition commonly encountered in the analysis of ordinary differential equations. Using this framework, a general FP theorem is proven, which extends the classical contraction in the setting of complete MSs. This result is further utilized, in conjunction with the characterization of weakly inward mappings, to derive several FP theorems in Banach spaces. These efforts led to the development of FP results for single-valued mappings in complete b-MSs [44, 45].

Nadler [46] is regarded as the founder of set-valued contractions and played a foundational role in developing FP results for MMs. Although the concept of MMs appeared in the early 20th century, Nadler formally introduced the idea of multi-valued contractive mappings and proved two important FP theorems. The first theorem is a generalization of the Banach contraction principle, demonstrating that a multi-valued contraction defined on a complete MS with values in non-empty, closed, and bounded subsets has a FP.

The second theorem extends Edelstein's result by addressing compact, set-valued contractions. Over time, the theory was extended to multivalued or set-valued mappings [47]. Reich [48] contributed to the theory by analyzing generalized contractions and set-valued mappings, particularly in normed spaces, and studied their FP properties under weaker assumptions. Rhoades [49] provided a systematic comparison of various generalized contractions, such as Ciric, Kannan, and Chatterjea-type contractions, helping to classify and unify these generalizations. Chatterjea [50] introduced a novel contractive condition where the contraction depends on the average of cross distances between image and pre-image points. Kannan [51] offered another generalization of Banach's theorem by formulating a contraction condition based on the average of distances from the image point to both arguments, resulting in what is now known as Kannan's FP theorem. This contraction condition is:

$$\mathsf{d}(\Re_1\zeta_1, \Re_1\zeta_2) \leq \lambda\{\mathsf{d}(\zeta_1, \Re_1\zeta_1) + \mathsf{d}(\zeta_2, \Re_1\zeta_2)\}, \text{ for all } \zeta_1, \zeta_2 \in \mathsf{G} \text{ and } 0 < \lambda < \frac{1}{2}.$$

In 2008 Berinde and Pacurar [52] introduced almost contractions which form a class of generalized contractions that includes several contractive type mappings like usual contractions, Kannan mappings, and Zamfirescu mappings. Rehman et al. [53] introduced some new type of multivalued contraction maps on H-cone metric and proved some FP and common FP theorems in the setting of cone MSs.

Mohammadi et al. [54] present some FP results for a class of nonexpansive self-mappings and MMs in the framework of b-MS . In 2012, Wardowski [55] introduced a notable generalization of the Banach contraction, known as the $\mathtt{F}$-contraction. This concept gained significant attention, and in 2013, Sagroia et al. [56] established FP results for $\mathtt{F}$-contractions, along with applications to integral equations.

Over time, the $\mathtt{F}$-contraction has been generalized in various ways. One significant generalization is the $(\alpha, \mathtt{F})$-contractive mapping, first introduced by Kamran et al. [32] in 2016 within the framework of b-MSs for single-valued mappings. This concept was later extended to MMs by Hussain et al. [57] in 2017.

Further, Batul et al. [58] proved FP results using $(\alpha^* - \mathtt{F})$ contractions in b-MSs. Moreover, Rasham et al. [59] established FP results for a multivalued dominated $\mathtt{F}$-contraction on a closed ball in a complete b-MS, utilizing the first condition of Wardowski's $\mathtt{F}$-contraction.

In this context Naz and Batul [60] introduced the Hardy Rogers contraction of the Nadler type by relaxing two conditions of Wardowski's mapping and proved several FP results In an orbitally complete b-MS, Nashine et al. [61] recently established FP results for set valued mappings that satisfy the Wardowski-Feng-Liu-type contraction for orbitally lower semi-continuous (LSC) functions.

Rasham et al. [62] introduced the advanced Nashine Wardowski Feng-Liu type contraction on orbitally complete b-MSs and proved several FP results on such spaces.

In [63] Branciari proved some $\mathcal{FP}$ results for mappings that satisfy integral-type contractive conditions.

Branciari investigated a self-mapping $\Re_1$ on G that satisfied the contractive requirements of the form

$$\int_0^{\mathsf{d}(\Re_1(\zeta_1),\Re_1(\zeta_2))} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \lambda \int_0^{\mathsf{d}(\zeta_1,\zeta_2)} \triangle(\varsigma)\mathrm{d}(\varsigma).$$

for any $\lambda \in (0,1)$ and $\zeta_1, \zeta_2 \in \mathsf{G}$, given a MS (G, d), where $\triangle : [0,\infty) \longrightarrow [0,\infty)$ is Lebesgue integrable and summable on each compact subset of $[0,\infty)$ and $\int_0^{\mathrm{t}} \triangle(\varsigma)\mathrm{d}(\varsigma) > 0$ for each $\mathrm{t} > 0$.

Jleli and Samet [64] introduce a new type of contraction known as Θ-contraction and establish a new FP theorem for such maps on the setting of generalized MSs. Later on, motivated by the idea of Samet et al. [64], Ameer et al. [65] presented the Θ-contraction by introducing extra condition in the settings b-MS.

1.2 Background of Fuzzy Sets Theory

The concept of fuzzy mathematics began with the pioneering work of Zadeh in 1965, who introduced the idea of FSs as a generalization of classical sets (see [66]). In contrast to traditional sets, where an element either belongs or does not belong to the set, FSs allow elements to have varying levels of membership. This is represented by a membership function that assigns values between 0 and 1.

With the rapid growth of fields such as artificial intelligence and neural networks, fuzzy logic has emerged as an essential tool for managing uncertainty and complex systems [67, 68]. This has also led to new and promising applications in areas such as chemical engineering, where fuzzy approaches offer practical solutions for complex problems.

Distance measurement plays a vital role in numerous domains, including remote sensing, data mining, and pattern recognition [69]. However, challenges arise when such measurements must be made under conditions of uncertainty or imprecision. In these contexts, the ambiguity of the available information can hinder logical decision-making. To arrive at well-informed conclusions [70], it is often essential to combine data with human judgment, experience, and expertise [71].

This necessity has led to the use of fuzzy numbers, which replace exact numerical values to better handle uncertainty. Although fuzzy numbers inherently involve imprecision, they frequently offer a more meaningful and flexible interpretation than traditional crisp values. Building on this foundation, a wide range of mathematical theories have been extended by incorporating the concepts of FSs and membership functions [72, 73]. For instance, Heilpern [74] introduced fuzzy mapping(FM) and established a FP result for fuzzy contraction mapping, as an extension of the Banach contraction principle in metric linear spaces. Azam et al. [75] established common FP theorems for FMs under a φ-contraction condition on a MS with the d_∞-metric which is induced by the Hausdorff metric on the family of FSs. Mohammadi et al. [76] introduced generalized α-η-fuzzy contractive mappings and generalized β-ζ-fuzzy establishing the existence of FPs for such mappings. Shamas et al. [77] explored some rational type coincidence point and derived common FP results for rational type weakly compatible self mappings in fuzzy MS. Rehman et al. [78] introduced the concept of MMs in fuzzy cone MSs, proving fundamental lemmas, defining a Hausdorff metric, and establishing FP results for set-valued fuzzy cone-contraction and multivalued fuzzy cone-contraction mappings. Additionally, Shagari et al. [79] extended the notions of soft set and fuzzy soft set and their applications to other domains. As a contribution to FSs theory Ameer et al. [80] introduced a significant advancement in 2020 by developing a new approach to FMs within complete MSs. Their work demonstrated the existence of α-fuzzy common FPs for a pair of FMs based on generalized almost (P, ψ)-contractive conditions. In 2022 Kanwal et al. [81] establish and prove some new common fuzzy FP theorems for FS-valued mappings involving Θ-contractions in a complete MS.

Recently, Azmi [82] introduced two new types of generalized contraction mappings in double controlled-MSs, namely, the Θ-double controlled contraction mapping and the Ciric-Reich-Rus-type-Θ-double controlled contraction mapping. He established the existence and uniqueness of the FP of these mappings and substantiate his results by providing an application.

1.3 Dissertation Contribution

In this dissertation, we extend the concept to the framework of b-MSs by introducing a (P, ψ)-type almost contraction tailored for FMs. This generalized framework enables the derivation of novel fuzzy FP results in complete b-MSs. To validate our findings, we include an example that satisfies the required conditions and illustrates the applicability of the main result. Furthermore, we address a second-order nonlinear boundary value problem by translating it into a fuzzy FP formulation and applying our established results. Several corollaries naturally emerge from the main theorem, reinforcing the significance of the proposed framework. These results are the extension of the results presented by Ameer et al. [80]. The published form of these results is available in [83] as "Fixed point results via fuzzy mappings in b-metric spaces and an application to differential equations"

Next we introduce two distinct classes of contraction mappings within the framework of double controlled-MS, with a particular focus on Θ-type contractions. The first category is identified as the Θ-fuzzy double controlled contraction mapping (Θ-FDCCM), while the second is known as the Θ-fuzzy almost generalized double controlled contraction mapping (Θ-FAGDCCM). For both classes, we prove the existence and uniqueness of their FPs and include illustrative examples to support the theoretical results.

In addition, we verify the effectiveness of these mappings by establishing a solution to a nonlinear differential equation. The developed theorems further lead to several corollaries, demonstrating that our findings extend and generalize the earlier results obtained by Azmi [82].

The published form of some of these results can be seen in [84] as "Novel Θ-Fuzzy-Contraction Mappings and Existence of Solution of Nonlinear Differential Equations". Motivated by the idea presented in [81] we develop and demonstrate several new, fuzzy FP theorems for FS-valued mappings using integral type Θ-contractions in the framework of b-MSs.

Numerous FP results are presented by using this new class defined on b-MSs. Appropriate illustrations are provided to support the results proved. Finally, an application to stochastic Volterra integral equations has been provided to strengthen our findings reliability.

Additionally, our work extends the findings of Rasham et al. [62] on α-dominated multivalued and orbitally LSC mappings in orbital controlled MSs. This extension is achieved by introducing an extended rational-type advanced Nashine–Wardowski–Feng–Liu contraction. We also derive new FP results for two α-dominated MMs in ordered orbitally complete controlled MSs.

These results are reinforced with illustrative examples and further applied to systems of fractional differential equations and nonlinear integral equations to confirm the existence of solutions. The published form of some of these results can be seen in [85] as "Existence of solutions to integral and fractional differential equations via common fixed points of extended contractions"

1.4 Organization of Dissertation

The rest of the dissertation is organized as follows:

(i): In Chapter 2, some basic definitions of abstract spaces and examples are stated.

(ii): In Chapter 3, we introduced the concept of an almost generalized contraction in the framework of b-MS. The primary objective of this chapter is to extend the results presented in [80] by employing the notion of an almost generalized contraction within b-MS. To accomplish this, the following challenges are addressed:

(i): The contraction inequality is refined by incorporating the parameter of the b-metric into its formulation.

(ii): To establish the Cauchyness of the iterative sequence, a previously established lemma from [86] is utilized.

Last part of chapter comprises a brief conclusion of our work. All the work of this chapter is published in [83].

(iii): In chapter 4, we introduces two distinct types of contraction mappings within the framework of double controlled-MS, with an emphasis on Θ-type contractions. The first is termed the Θ-`FDCCM`, while the second is referred to as the Θ-`FAGDCCM`. For both classes of mappings, the existence of FP is established. To support and validate the main findings, illustrative example is presented. Additionally, a series of corollaries are derived from the main theorems, demonstrating that numerous existing FP results can be viewed as special cases of the proposed results. Finally, the practical relevance of the theory is highlighted through its application to a second-order nonlinear boundary value problem, where the existence of a solution is confirmed using the developed FP results. The final part of this chapter presents a concise conclusion, summarizing the key contributions and findings of the study. All the work of this chapter is published in [84].

(*iv*): In chapter 5 we have introduced the notion of fuzzy Θ-type generalized almost contraction mappings(F_ΘGAC). The main aim of this chapter is to obtain the result of [81] by using F_ΘGAC. All the findings of this chapter are submitted for possible publication.

(*v*): Chapter 6 contains some new definitions, FP results and example, providing clarity and insight. To further validate our findings, we apply them to prove the existence of solution to a nonlinear Volterra-type integral equation and a fractional differential equation. We basically generalize the findings presented by Rasham et al. [62]. All the work of this chapter is published in [85].

Chapter 2

Core Concepts

This foundational chapter establishes the essential conceptual framework necessary for this dissertation, providing a solid groundwork for subsequent discussions. The key themes presented here are crucial for a clear and comprehensive exposition of the research.

The primary objective of this chapter is to synthesize relevant scholarly literature, focusing on core concepts and principles without delving into formal proofs or technical derivations. This chapter is distributed in different sections for better presentation as follows:

(i): The first section discusses the concepts of FSs.

(ii): The second section covers the idea of **partial ordering**.

(iii): The third section presents a brief **history of metric spaces**.

(iv): The fourth section introduces various **generalizations of metric spaces**.

(v): The fifth section includes the **Banach contraction principle** and its **generalizations**.

2.1 Fuzzy Sets

Zadeh [66] laid the foundation of FSs as described below:

Definition 2.1.1. A FS in G is a function with domain G and values in $[0,1]$. If F is FS and $\zeta \in \mathrm{F}(\mathsf{G})$, then the function values $\mathrm{F}(\zeta)$ is called grade of membership of ζ in F and $\mathcal{F}(\mathsf{G})$ denotes the collection of all FSs in G.

Example 2.1.2. Consider $\mathsf{G} = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ and $\mathrm{F} : \mathsf{G} \to [0,1]$ defined as:

$\mathrm{F}(\zeta_1) = 0$,

$\mathrm{F}(\zeta_2) = 0.5$,

$\mathrm{F}(\zeta_3) = 0.2$,

$\mathrm{F}(\zeta_4) = 1$.

Then F is a FS on G. This FS can also be written as follows:

$$\mathrm{F} = \{(\zeta_1, 0), (\zeta_2, 0.5), (\zeta_3, 0.2), (\zeta_4, 1)\}.$$

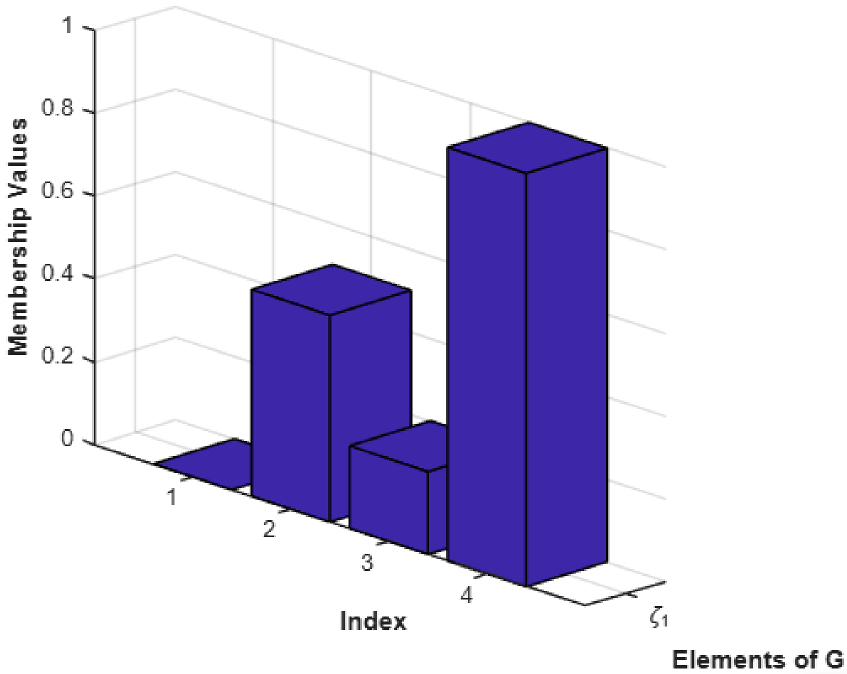

FIGURE 2.1: Graph of FS

Definition 2.1.3. Let $\mathsf{G} \neq \emptyset$, and let d be a metric on G. The α-level set of $\mathcal{F}$, symbolized as $\mathcal{F}_\alpha$, is formulated as

$$[\mathcal{F}]_\alpha = \left\{\zeta \in \mathsf{G} : \mathcal{F}(\zeta) \geq \alpha\right\}, \text{ if } \alpha \in (0,1],$$
$$[\mathcal{F}]_0 = \overline{\left\{\zeta \in \mathsf{G} : \mathcal{F}(\zeta) > 0\right\}},$$

where $\overline{\mathcal{F}}$ indicates the closure of $\mathcal{F}$. [75]

Example 2.1.4. Let $\mathsf{G} = \{1, 2, 3\}$ with usual metric d and $\mathrm{F} = \{(1, 0.3), (2, 0.7), (3, 0.5)\}$ be a FS in G.
If $\alpha = 0.3$ then $[\mathrm{F}]_{0.3} = \{1, 2, 3\}$.
If $\alpha = 0.6$ then $[\mathrm{F}]_{0.6} = \{2\}$.
If $\alpha = 0.4$ then $[\mathrm{F}]_{0.4} = \{2, 3\}$.

Definition 2.1.5. Let $\mathsf{G} \neq \emptyset$ and Y be a MS. A mapping F is called FM if F is a mapping from G into $\mathcal{F}(\mathsf{Y})$. [87]

Example 2.1.6. Consider the intervals $\mathsf{G} = [-10, 10]$ and $Y = [-5, 5]$. A mapping $F : \mathsf{G} \longrightarrow \mathcal{F}(Y)$ defined by

$$F(\zeta)(\mathsf{y}) = \frac{\zeta^2 + \mathsf{y}^2}{130}$$

is a FM. The graphical representation of $F(\zeta)(\mathsf{y})$, illustrating the possible membership values of y in $F(\zeta)$, is depicted in Figure 2.2.

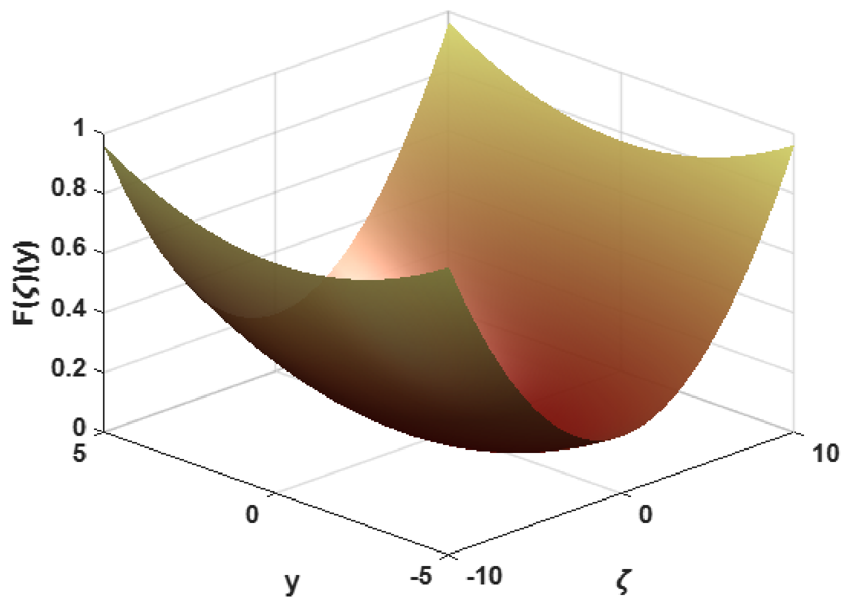

FIGURE 2.2: Graph of FM F

Example 2.1.7. Consider the intervals $\mathsf{G} = [0, 20]$ and $\mathsf{Y} = [0, 15]$. A mapping $F : \mathsf{G} \longrightarrow \mathcal{F}(\mathsf{Y})$ defined by

$$F(\zeta)(\mathsf{y}) = \frac{\zeta + \mathsf{y} + \zeta\mathsf{y}}{360}$$

is a FM, where F's output values range between 0 and 1 for all input pairs $(\zeta, \mathsf{y}) \in \mathsf{G} \times Y$.

The graphical representation of $F(\zeta)(y)$, illustrating the possible membership values of y in $F(\zeta)$, is depicted in Figure 2.3.

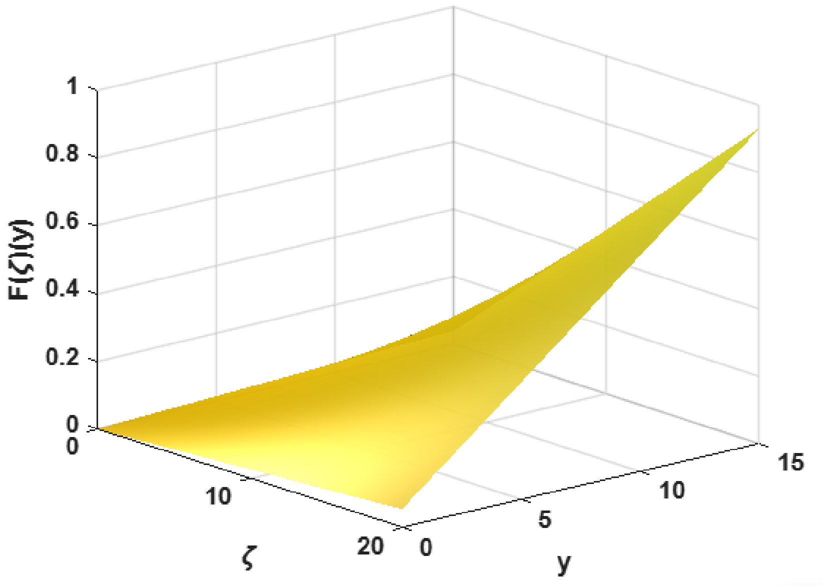

FIGURE 2.3: Graph of FM F

2.2 Partial Ordering

Definition 2.2.1. A partially ordered set is a set G equipped with a partial order, denoted by $\preceq$. This is a binary relation on G that satisfies the following conditions:

(PO_1) : $\zeta_1 \preceq \zeta_1$ for every $\zeta_1 \in \mathsf{G}$.

(PO_2) : $\zeta_1 \preceq \zeta_2$ and $\zeta_2 \preceq \zeta_1$, then $\zeta_1 = \zeta_2$.

(PO_3) : $\zeta_1 \preceq \zeta_2$ and $\zeta_2 \preceq \zeta_3$, then $\zeta_1 \preceq \zeta_3$.[88]

Definition 2.2.2. A totally ordered set, often known as a chain, is a partially ordered set in which each pair of elements is comparable. To put it another way, a partially ordered set without any incomparable elements is called a chain.[88]

Remark 2.2.3. Every totally ordered set is also a partially ordered set, while the opposite is not true.[88]

Example 2.2.4. The following examples will elaborate the above idea:

(*i*): The set $\mathbb{R}$ constitutes a totally ordered set with respect to the standard order $\leq$.

(*ii*): Assume $\mathsf{G} \neq \emptyset$ and $\mathsf{P}(\mathsf{G})$ is a power set of G with a relation $\preceq$ is given by the inclusion relation, that is $C \preceq D$ if $C \subseteq D$, where $C, D \in \mathsf{P}(\mathsf{G})$. One can easily check that $\preceq$ is a partial order and $\mathsf{P}(\mathsf{G})$ is partially ordered set.

(*iii*): Consider

$$\mathsf{G} = \mathbb{R} \times \mathbb{R} = \{(\zeta_1, \zeta_2) : \zeta_1, \zeta_2 \in \mathbb{R}\}.$$

Define an order $\preceq$ on the set G as follows:

$$\zeta_1 \leq \zeta_3, \zeta_2 \leq \zeta_4 \Leftrightarrow (\zeta_1, \zeta_2) \preceq (\zeta_3, \zeta_4).$$

Here $\leq$ is the usual order on the elements of $\mathbb{R}$. Then it can be seen easily that $\preceq$ is a partial order on the given set G or G is a partially ordered set.

Definition 2.2.5. Let $\emptyset \neq \mathsf{G} \subseteq \mathbb{R}$ and a real valued function $\Re_1 : \mathsf{G} \to \mathbb{R}$. Then for $\delta > 0$ the limit supremum of mapping $\Re_1$ and limit infimum of mapping $\Re_1$ are defined in the following way:

$$\lim_{\zeta_2 \to \zeta_1} \sup \Re_1(\zeta_2) = \begin{cases} \sup\{\Re_1(\zeta_2) : |\zeta_1 - \zeta_2| < \delta\}; & \text{if the supremum exists,} \\ \infty; & \text{otherwise,} \end{cases}$$

$$\lim_{\zeta_2 \to \zeta_1} \inf \Re_1(\zeta_2) = \begin{cases} \inf\{\Re_1(\zeta_2) : |\zeta_1 - \zeta_2| < \delta\}; & \text{if the infimum exists,} \\ \infty; & \text{otherwise.} \end{cases} \quad [89]$$

Example 2.2.6. Let $\{\zeta_t\}$ be the sequence $\left\{\frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, ...\right\}$. Now we consider that the $\{a_t\}$ and $\{b_t\}$ are the sequence of the infimum and supremum of the sub-sequences:

$$\{a_t\} = \left\{-\frac{1}{2t+1}\right\}, \quad \text{where}, t \in \mathbb{N}$$

and

$$\{b_t\} = \left\{\frac{1}{2t}\right\}, \qquad \text{where}, \quad t \in \mathbb{N}.$$

The infimum of the sequence $\{\zeta_t\}$ is the greatest lower bound of all subsequences of $\{a_t\}$, and the supremum of $\{\zeta_t\}$ is the least upper bound of all subsequences of $\{b_t\}$.

Therefore,

$$\lim_{t\to\infty} \sup \zeta_t = \lim_{t\to\infty} \left\{\frac{1}{2t}\right\} = 0,$$

and

$$\lim_{t\to\infty} \inf \zeta_t = \lim_{t\to\infty} \left\{-\frac{1}{2t+1}\right\} = 0.$$

Thus, the limit of both the sequences i. e. $\{a_t\}$ and $\{b_t\}$ is 0 as $t \to \infty$, which is also the limit of sequence $\{\zeta_t\}$.

$$\lim_{t\to\infty} \zeta_t = 0.$$

2.3 Metric Spaces

The idea of using abstract space in a systematic manner is first given in 1906 by Fréchet [7] and it is justified by its usefulness in different fields of mathematics.

Definition 2.3.1. A MS is a pair (G, d), where $\mathsf{G} \neq \emptyset$ and d is a metric on G, that is, a function defined on $\mathsf{G} \times \mathsf{G}$ such that $\forall\ \zeta_1, \zeta_2, \zeta_3 \in \mathsf{G}$ we have:

(M_1): d is real-valued and nonnegative.

(M_2): $\mathsf{d}(\zeta_1, \zeta_2) = 0 \Leftrightarrow \zeta_1 = \zeta_2$.

(M_3): $\mathsf{d}(\zeta_1, \zeta_2) = \mathsf{d}(\zeta_2, \zeta_1)$.

(M_4): $\mathsf{d}(\zeta_1, \zeta_2) \leq \mathsf{d}(\zeta_1, \zeta_3) + \mathsf{d}(\zeta_3, \zeta_2)$. [10]

Example 2.3.2. Consider $\mathsf{G} = \ell^\infty$, the set of all bounded sequences of complex numbers, and the distance function $\mathsf{d} : \mathsf{G} \times \mathsf{G} \to \mathbb{R}$ defined as:

$$\mathsf{d}(\zeta_1, \zeta_2) = \sup_{n \in \mathbb{N}} |\zeta_{1n} - \zeta_{2n}| \quad \text{where } \zeta_1, \zeta_2 \in \mathsf{G}.$$

Here, d satisfies conditions (M_1)-(M_4).

Thus, (G, d) is a MS.

Definition 2.3.3. A sequence $\{\zeta_t\}$ in a MS (G, d) is called convergent if there is a point $\zeta \in \mathsf{G}$ such that

$$\lim_{t \to \infty} \mathsf{d}(\zeta_t, \zeta) = 0. \text{ [10]}$$

Let us recall from the fundamentals of real analysis that a sequence $\{\zeta_t\}$ of real numbers is convergent in the real line $\mathbb{R}$ as well as a sequence $\{\zeta_t\}$ of complex numbers is convergent in the complex plane $\mathbb{C}$ if and only if it satisfies the Cauchy criterion for convergence, that is, for every number $\epsilon > 0$, $\exists\ N = N(\epsilon)$ such as

$$|\zeta_t - \zeta_s| < \epsilon; \quad \text{for;} \quad t, s \geq N.$$

Generally, this is not true as there are Cauchy sequences which do not converge.

Definition 2.3.4. A sequence ζ_n in a MS (G, d) is called Cauchy if for every $\epsilon > 0\ \exists\ N = N(\epsilon)$ such that

$$\mathsf{d}(\zeta_s, \zeta_t) < \epsilon \quad \text{for every } s, t > N. \text{[10]}$$

Definition 2.3.5. A MS (G, d) is called complete if every Cauchy sequence in G converges to a point in G. [10]

Examples of complete MSs include the real numbers and complex numbers with the usual metric, as well as any set equipped with a discrete metric, whereas examples of incomplete spaces include the nonzero real numbers, the rationals, and open intervals under the usual metric.

2.4 Fixed Point

Definition 2.4.1. A point $\zeta \in \mathsf{G}$ is called a FP of the mapping $\Re_1 : \mathsf{G} \to \mathsf{G}$ if

$$\Re_1(\zeta) = \zeta,$$

that is, the image of ζ under $\Re_1$ equals ζ itself. [10]

For real-valued functions, FPs correspond to the points where the graph of the function $\Re_1\zeta = \eta$, intersects the identity line $\zeta = \eta$.

Depending on the nature of the mapping, there may exist a single FP, multiple FPs, or none at all.

Some illustrative examples of FPs are shown below:

Example 2.4.2. Let $\mathsf{G} = \mathbb{R}$. The mapping $\Re_1 : \mathsf{G} \to \mathsf{G}$ defined by

$$\Re_1(\zeta) = \zeta^2 + 2$$

has no FP in G.

Geometrically it means that the graph of $\Re_2(\zeta) = \zeta$ never intersects the graph of $\Re_1(\zeta) = \zeta^2 + 2$ (see figure 2.4).

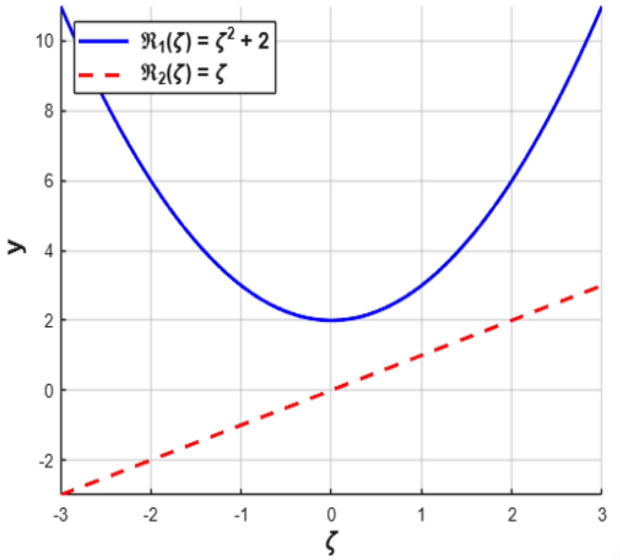

FIGURE 2.4: A mapping having no FP.

Example 2.4.3. Let $\mathsf{G} = \mathbb{R}$ and defined $\Re_1 : \mathsf{G} \to \mathsf{G}$ by

$$\Re_1 \zeta = \zeta + \sin(\zeta) \ \forall \ \zeta \in \mathbb{R},$$

possesses infinite many FPs. The graphical representation is given in Figure 2.5.

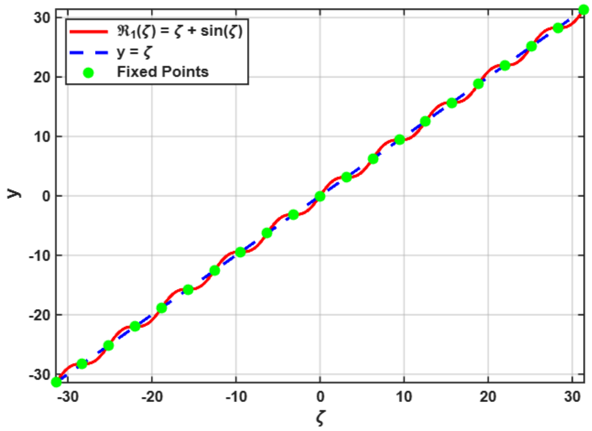

FIGURE 2.5: A mapping having infinite FPs.

Example 2.4.4. Let $\mathsf{G} = \mathbb{R}$ and defined $\Re_1 : \mathsf{G} \to \mathsf{G}$ by $\Re_1 \zeta = \zeta^3 \ \ \forall \ \zeta \in \mathbb{R}$. It is clear from Figure 2.6 that 0, 1and -1 are three FPs.

Geometrically it means that the graph of $\Re_2(\zeta) = \zeta$ intersects the graph of $\Re_1(\zeta) = \zeta^3$ at three points 0, 1 and -1.

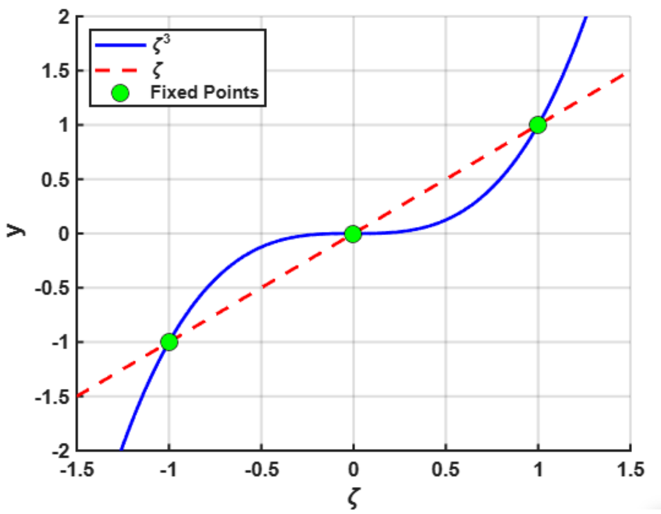

FIGURE 2.6: A mapping having three FPs.

Definition 2.4.5. Let $\mathsf{G} = (\mathsf{G}, \mathsf{d})$ and $\mathsf{Y} = (\mathsf{Y}, \mathsf{d}_1)$ be MSs. A mapping $\Re_1 : \mathsf{G} \to \mathsf{Y}$ is said to be continuous at a point $\zeta_0 \in \mathsf{G}$ if for every $\epsilon > 0$ $\exists$ a $\delta > 0$

such that

$$\mathsf{d}_1(\Re_1\zeta, \Re_1\zeta_0) < \epsilon$$

whenever

$$\mathsf{d}(\zeta, \zeta_0) < \delta$$

$\forall\ \zeta \in \mathsf{G}$. $\Re_1$ is continuous if it is continuous at all points of G. [10]

Example 2.4.6. Consider $\mathsf{G} = [0, 1]$ equipped with the usual metric.
Define a mapping $\Re_1 : \mathsf{G} \to \mathbb{R}$ by

$$\Re_1(\zeta) = 2\zeta + 1,$$

$\forall\ \zeta \in [0, 1]$. Then $\Re_1$ is continuous on G.

Definition 2.4.7. Let $\Re_1 : \mathsf{D} \to \mathbb{R}$ and let $\zeta_0 \in \mathsf{D}$. We say that $\Re_1$ is LSC at ζ_0 if for every $\epsilon > 0$ $\exists$, a $\delta > 0$

such that

$$\Re_1(\zeta_0) - \epsilon \leq \Re_1(\zeta)$$

$\forall\ \zeta \in \mathsf{B}(\zeta_0, \delta) \cap \mathsf{D}$.

Similarly, the mapping $\Re_1$ is called upper semi-continuous at ζ_0 if, for every $\epsilon > 0$, $\exists\ \delta > 0$

such that

$$\Re_1(\zeta) \leq \Re_1(\zeta_0) + \epsilon,$$

for all $\zeta \in \mathsf{B}(\zeta_0, \delta) \cap \mathsf{D}$.

Thus, $\Re_1$ is continuous at ζ_0 precisely when it is both LSC and upper semi-continuous at that point. [90]

The following examples illustrate the above concepts:

Example 2.4.8. Let $\mathsf{G} = \mathbb{R}$ and $\mathfrak{R}_1 \colon \mathsf{G} \to \mathsf{G}$ be a mapping defined as

$$\mathfrak{R}_1(\zeta) = \begin{cases} \frac{1}{\zeta}; & \text{if } \zeta < 0, \\ 0; & \text{if } \zeta = 0, \\ -\frac{1}{\zeta}; & \text{if } \zeta > 0. \end{cases}$$

If $\zeta_0 \longrightarrow 0$, then

$$\begin{aligned} \mathfrak{R}_1(\zeta_0) \to -\infty &< 0 \\ &= \mathfrak{R}_1(0). \end{aligned}$$

The right and left limit of the mapping is $-\infty$ which is different from the value of the mapping that is 0.

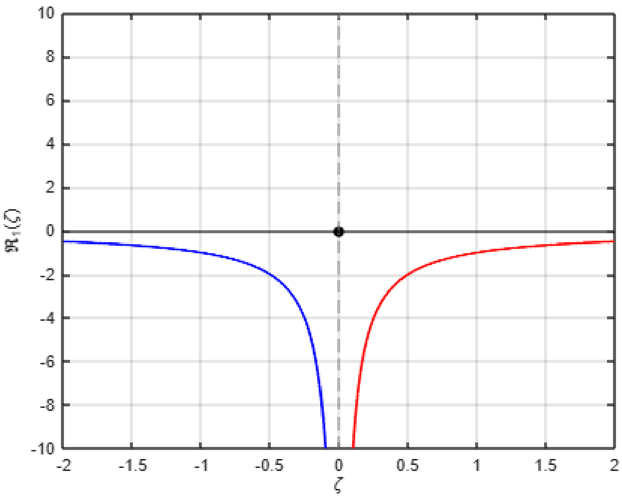

FIGURE 2.7: This graph reveals that $\mathfrak{R}$ is an upper semi-continuous point at $\zeta = 0$.

Example 2.4.9. Consider $\mathsf{G} = \mathbb{R}$, and let $\mathfrak{R}_1 : \mathsf{G} \to \mathsf{G}$ be a function given by

$$\mathfrak{R}_1(\zeta) = \begin{cases} \zeta^2; & \text{if, } \zeta \neq 0, \\ -1; & \text{if, } \zeta = 0. \end{cases}$$

This function is LSC at the point $\zeta = 0$.

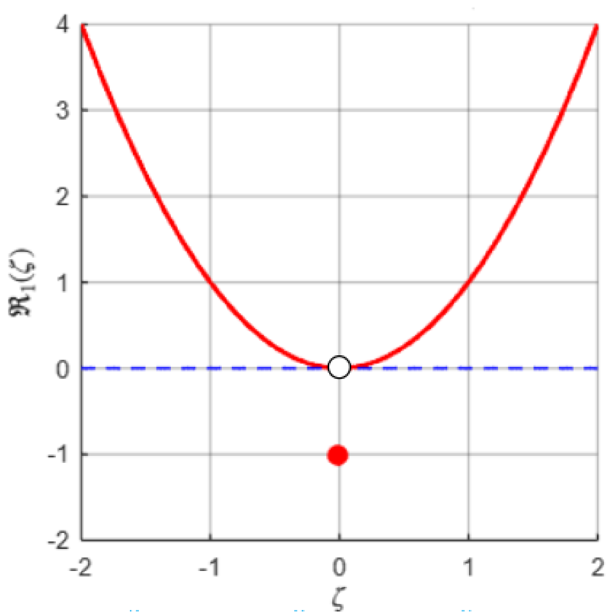

FIGURE 2.8: The graph indicates that $\zeta = 0$ serves as a point of LSC.

Example 2.4.10. Consider $\mathsf{G} = \mathbb{R}$, and let $\Re_1 : \mathsf{G} \to \mathsf{G}$ be a function given by

$$\Re_1(\zeta) = \begin{cases} -\zeta^2; & \text{if, } \zeta \neq 0, \\ 1; & \text{if, } \zeta = 0. \end{cases}$$

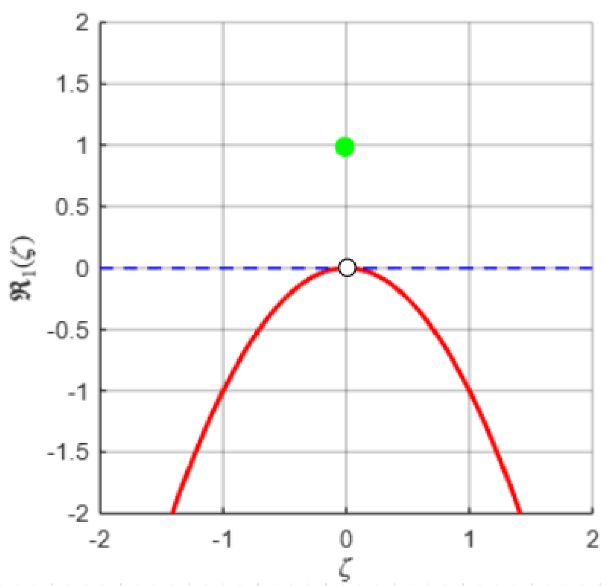

FIGURE 2.9: The graph indicates that $\zeta = 0$ serves as a point of upper semi-continuous point.

The function is upper semi-continuous at $\zeta = 0$. To define the concept of $\Re_1$-orbitally continuous maps, the following concept is required:

Definition 2.4.11. Let $\Re_1 \colon \mathsf{G} \to \mathsf{G}$ and for some $\zeta_0 \in \mathsf{G}$,

$$\mathsf{O}_{\Re_1}(\zeta_0) = \{\zeta_0, \Re_1\zeta_0, {\Re_1}^2\zeta_0, ...\}$$

be the orbit of ζ_0. [24]

Example 2.4.12. Consider a set $\mathsf{G} = [0, 2]$ with usual metric. Define $\Re_1 \colon \mathsf{G} \to \mathsf{G}$ as:

$$\Re_1(\zeta) = \begin{cases} \dfrac{\zeta}{2}; & \text{if } \ \zeta \in [0, 1), \\ \\ \dfrac{1+\zeta}{2}; & \text{if } \ \zeta \in [1, 2). \end{cases}$$

Assuming $\zeta_0 = \dfrac{1}{4} \in \mathsf{G}$ then we have

$$\begin{aligned} \mathsf{O}_{\Re_1}(\zeta_0) &= \{\zeta_0, \Re_1\zeta_0, {\Re_1}^2\zeta_0, ...\} \\ &= \left\{\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...\right\} \\ &= \left\{\frac{1}{2^t}, \ \ t \geq 2, \ \ t \in \mathbb{Z}^+\right\}. \end{aligned}$$

Let $\zeta_0 = \dfrac{3}{2} \in \mathsf{G}$ then we have

$$\begin{aligned} \mathsf{O}_{\Re_1}(\zeta_0) &= \{\zeta_0, \Re_1\zeta_0, {\Re_1}^2\zeta_0, ...\} \\ &= \left\{\frac{3}{2}, \frac{5}{4}, \frac{9}{8}, ...\right\}. \end{aligned}$$

Example 2.4.13. Consider a set $\mathsf{G} = [-2, 2] \times [-2, 2]$ and define a self mapping $\Re_1 \colon \mathsf{G} \to \mathsf{G}$ on the set G in the following way:

$$\Re_1(\zeta_0) = \Re_1(\zeta_1, \zeta_2) = \begin{cases} (0.5\zeta_1, 0.5\zeta_2); & \text{if } \ \zeta_1, \zeta_2 \geq 0, \\ \\ (2, 0); & \text{otherwise.} \end{cases}$$

Obviously, the mapping $\Re_1$ is not continuous at $(0, 0) \in \mathsf{G}$.

Assuming $\zeta_0 = (\zeta_1, \zeta_2) \in \mathsf{G}$ such that $\zeta_2 < 1, 0 < \zeta_1$, so we obtain

$$\mathsf{O}_{\Re_1}(\zeta_0) = \{\zeta_0, \frac{\zeta_0}{2}, \frac{\zeta_0}{4}, ...\}.$$

Definition 2.4.14. A function $\Re_1 : \mathsf{G} \to \mathbb{R}$ defined on a non-empty set G is called $\Re_1$-orbitally LSC at a point $v \in \mathsf{G}$ if every sequence $\zeta_t \subset \mathsf{O}_{\Re_1}(\zeta)$ with $\zeta_t \to v$ satisfies

$$\Re_1(v) \leq \lim_{t\to\infty} \inf \Re_1(\zeta_t). \text{ [24]}$$

MMs are widely used in the literature. For the purposes of this study, the following definition is fundamental.

Definition 2.4.15. A mapping $\Re_1 : \mathsf{G} \to \mathrm{P}(\mathsf{Y})$ is called a multi-valued map if, for every element $\zeta_1 \in \mathsf{G}$, the set $\Re_1\zeta_1$ is a non-empty subset of Y. Equivalently, a MM $\Re_1$ from G to $\mathrm{P}(\mathsf{Y})$ can be viewed as a non-empty subset of the product space $\mathsf{G} \times \mathsf{Y}$. Thus, if $\Re_1 \subset \mathsf{G} \times \mathsf{Y}$ is non-empty, then $\Re_1$ is regarded as a multi-valued map, and the image of any $\zeta_1 \in \mathsf{G}$ under $\Re_1$ is denoted by $\Re_1\zeta_1$ and defined by

$$\Re_1\zeta_1 = \{\zeta_2 \in \mathsf{Y} : (\zeta_1, \zeta_2) \in \Re_1\} \subset \mathsf{Y},$$

where both G and Y are non-empty sets. [91]

Example 2.4.16. Let $\mathsf{G} = [0, 1]$ and $\mathrm{P}(\mathsf{G}) = \{\mathfrak{U} \subset \mathsf{G} : \mathfrak{U} \neq \emptyset\}$. Then the mapping $\Re_1 : \mathsf{G} \to \mathrm{P}(\mathsf{G})$ defined by

$$\Re_1\zeta = \left[0, \zeta^2\right]$$

is a MM. Its graph is given as:

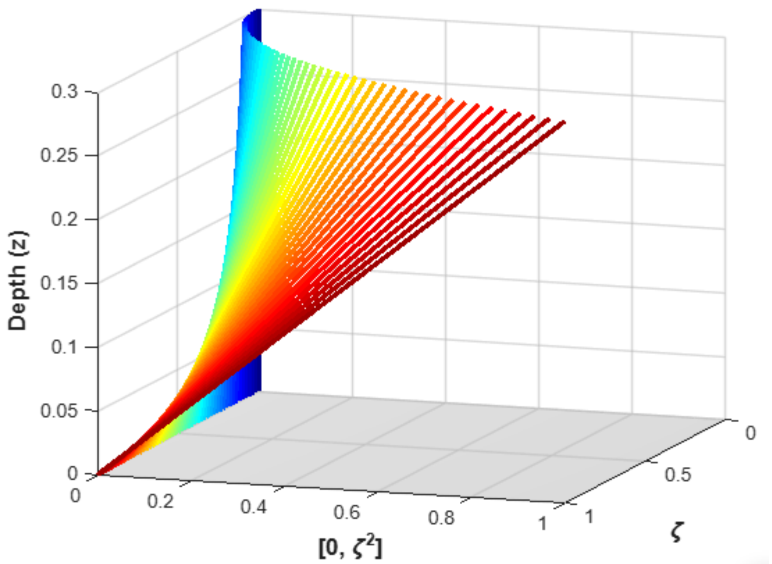

FIGURE 2.10: Graph of MM.

Example 2.4.17. Let $\mathsf{G} = [0,1]$ and consider the collection of closed and bounded subset of G as $\mathfrak{CB}(\mathsf{G})$.

Define a map $\Re_1 : \mathsf{G} \to \mathfrak{CB}(\mathsf{G})$ as

$$\Re_1\zeta = \begin{cases} [0,1]\,; & \text{if} \quad \zeta \neq \dfrac{3}{5}, \\ \left[\dfrac{3}{5}, 1\right]; & \text{if} \quad \zeta = \dfrac{3}{5}, \end{cases}$$

is a MM.

The graph of this mapping is shown in the following figure.

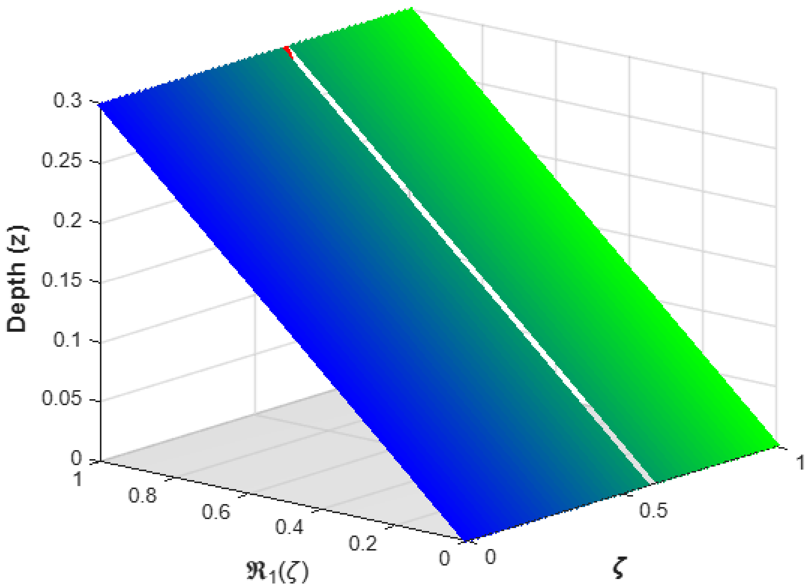

FIGURE 2.11: Graph of MM.

Definition 2.4.18. An element $\zeta \in \mathsf{G}$ for any non-empty set G, is said to be a FP for a mapping $\Re_1 : \mathsf{G} \to \mathrm{P}(\mathsf{G})$ if $\zeta \in \Re_1\zeta \subseteq \mathsf{G}$. [91]

Example 2.4.19. Let $\mathsf{G} = [0,1]$, define a mapping $\Re_1 : \mathsf{G} \to \mathrm{P}(\mathsf{G})$ by

$$\Re_1\zeta = [0, \zeta^2],$$

then 1 and 0 are FPs of the mapping $\Re_1$.

Definition 2.4.20. Let $\Re_1, \Re_2 : \mathsf{G} \to \mathrm{P}(\mathsf{G})$ be two MMs. A point $\zeta_0 \in \mathsf{G}$ is called a common FP of $\Re_1$ and $\Re_2$ if

$$\zeta_0 \in \Re_1\zeta_0 \cap \Re_2\zeta_0. \text{ [91]}$$

Example 2.4.21. Consider $\mathsf{G} = [0,1]$ and let $\zeta \in \mathsf{G}$. Now define $\Re_1, \Re_2 : \mathsf{G} \to \mathrm{P}(\mathsf{G})$ as

$$\Re_1\zeta = \left[0, \frac{\zeta}{4}\right]; \quad \forall \quad \zeta \in \mathsf{G}$$

and

$$\Re_2\zeta = \left[0, \frac{\zeta}{2}\right]; \quad \forall \quad \zeta \in \mathsf{G}$$

then it is simple to find that 0 is a common FP for mappings $\Re_1$ and $\Re_2$.

Definition 2.4.22. Let $\Re_1 : \mathsf{G} \longrightarrow \mathcal{F}(\mathsf{G})$ be a FM. We can define a set-valued map

$$\hat{\Re}_1 : \mathsf{G} \longrightarrow \mathfrak{CB}(\mathsf{G})$$

by

$$\hat{\Re}_1(\zeta_1) = \left\{\zeta_2 \in \mathsf{G} : \Re_1(\zeta_1)(\zeta_2) = \max_{\mathsf{t}\in\mathsf{G}} \Re_1(\zeta_1)(\mathsf{t})\right\}.$$

A point $\zeta^* \in \mathsf{G}$ is called a FP of the FM $\Re_1$ if

$$\Re_1(\zeta^*)(\zeta^*) \geq \Re_1(\zeta^*)(\zeta) \quad \forall\, \zeta \in \mathsf{G}. \text{ [92]}$$

Definition 2.4.23. Consider κ be the set of functions $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ so that

(κ_1): ψ is increasing;

(κ_2): For each sequence $\{\mathrm{t_n}\} \subset \mathbb{R}^+$, $\lim_{n\longrightarrow\infty} \psi(\mathrm{t_n}) = 0 \Leftrightarrow \lim_{\mathrm{n}\longrightarrow\infty} \mathrm{t_n} = 0$;

(κ_3): ψ is continuous. [80]

Definition 2.4.24. Consider υ be the set of functions $P : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

(υ_1): $\mathrm{t}_1 < \mathrm{t}_2 \Rightarrow P(\mathrm{t}_1) \leq P(\mathrm{t}_2)$;

(υ_2): $\lim_{\mathrm{n}\longrightarrow\infty} P^{\mathrm{n}}(\mathrm{t}) = 0 \quad \forall \;\; \mathrm{t} > 0$, P^{n} denotes the nth iterate of P.

such a P is called a comparison function. [93]

2.5 Generalization of Metric Space

In Euclidean spaces, the idea of distance between points extends naturally to a broader context, giving rise to the concept of a metric on an arbitrary nonempty set G. Over time, numerous researchers have proposed various generalizations of the classical MS framework. Notably, in 1989, Bakhtin [19] introduced the concept of a b-MS by using a real number $s \geq 1$ in the triangular inequality.

Definition 2.5.1. Assume $\mathsf{G} \neq \emptyset$ with $s \geq 1 \in \mathbb{R}$. A function $\mathsf{d}_b : \mathsf{G} \times \mathsf{G} \to [0, \infty)$ is a b-metric if it meets the following conditions $\forall\ \zeta_1, \zeta_2, \zeta_3 \in \mathsf{G}$:

(b_1): $\mathsf{d}_b(\zeta_1, \zeta_2) = 0 \ \Leftrightarrow\ \zeta_1 = \zeta_2$,

(b_2): $\mathsf{d}_b(\zeta_1, \zeta_2) = \mathsf{d}_b(\zeta_2, \zeta_1)$,

(b_3): $\mathsf{d}_b(\zeta_1, \zeta_2) \leq s\,[\mathsf{d}_b(\zeta_1, \zeta_3) + \mathsf{d}_b(\zeta_3, \zeta_2)]$.

The pair $(\mathsf{G}, \mathsf{d}_b)$ is called a b-MS. [19]

Example 2.5.2. Let $\mathsf{G} = \{0, 1, 2\}$. Consider a mapping $\mathsf{d}_b : \mathsf{G} \times \mathsf{G} \to [0, \infty)$ defined as

$$\mathsf{d}_b(\zeta_1, \zeta_2) = (\zeta_1 - \zeta_2)^2.$$

Then $(\mathsf{G}, \mathsf{d}_b)$ is a b-MS with $s = 2$.

Example 2.5.3. Let (G, d) be a MS. Define $\mathsf{d}_1 : \mathsf{G} \times \mathsf{G} \to \mathbb{R}$ by

$$\mathsf{d}_1(\zeta_1, \zeta_2) = \alpha_1 \mathsf{d}(\zeta_1, \zeta_2) + a(\mathsf{d}(\zeta_1, \zeta_2))^{\alpha_2},$$

where $\alpha_1 \geq 0$, $a > 0$, and $\alpha_2 > 1$.
Then $(\mathsf{G}, \mathsf{d}_1)$ is a b-MS with $s = 2^{\alpha_2 - 1}$, but d_1 is not a metric on G.

The notions of Cauchy sequences and convergence in a b-MSs are analogous to those in MSs.

Kamran et al. [24] generalized the concept of b-MS by introducing extended b-MSs, where the triangular inequality involves a function that depends on the specific points ζ_1 and ζ_2.

Definition 2.5.4. Let $\mathsf{G} \neq \emptyset$ and $\tilde{\theta} : \mathsf{G} \times \mathsf{G} \to [1, \infty)$ be a mapping. A function $\mathsf{d} : \mathsf{G} \times \mathsf{G} \to [0, \infty)$ is known as an extended b-metric if $\forall\, \zeta_1, \zeta_2, \zeta_3 \in \mathsf{G}$, it satisfies the following conditions:

(E_1): $\mathsf{d}_{\tilde{\theta}}(\zeta_1, \zeta_2) = 0 \Leftrightarrow \zeta_1 = \zeta_2$,

(E_2): $\mathsf{d}_{\tilde{\theta}}(\zeta_1, \zeta_2) = \mathsf{d}_{\tilde{\theta}}(\zeta_2, \zeta_1)$,

(E_3): $\mathsf{d}_{\tilde{\theta}}(\zeta_1, \zeta_3) \leq \tilde{\theta}(\zeta_1, \zeta_3)\, [\mathsf{d}_{\tilde{\theta}}(\zeta_1, \zeta_2) + \mathsf{d}_{\tilde{\theta}}(\zeta_2, \zeta_3)]$.

The pair $(\mathsf{G}, \mathsf{d}_{\tilde{\theta}})$ is called an extended b-MS. [24]

Example 2.5.5. Consider $\mathsf{G} = \{1, 2, 3\}$. Define mappings $\mathsf{d}_{\tilde{\theta}} : \mathsf{G} \times \mathsf{G} \to [0, \infty)$ by

$$\mathsf{d}_{\tilde{\theta}}(\zeta_1, \zeta_2) = (\zeta_1 - \zeta_2)^2$$

and $\tilde{\theta} : \mathsf{G} \times \mathsf{G} \to [1, \infty)$ as

$$\tilde{\theta}(\zeta_1, \zeta_2) = 2 + \zeta_1 + \zeta_2.$$

Here $\mathsf{d}_{\tilde{\theta}}$ satisfies the conditions $(E_1) - (E_3)$. Thus $(\mathsf{G}, \mathsf{d}_{\tilde{\theta}})$ is an extended b-MS.

Example 2.5.6. Consider $\mathsf{G} = \{1, 2, 3\}$. Define $\tilde{\theta} : \mathsf{G} \times \mathsf{G} \to [1, \infty)$ and $\mathsf{d}_{\tilde{\theta}} : \mathsf{G} \times \mathsf{G} \to [0, \infty)$ as

$$\tilde{\theta}(\zeta_1, \zeta_2) = 1 + \zeta_1 + \zeta_2,$$

and

$$\mathsf{d}_{\tilde{\theta}}(1, 1) = \mathsf{d}_{\tilde{\theta}}(2, 2) = \mathsf{d}_{\tilde{\theta}}(3, 3) = 0,$$

$$\mathsf{d}_{\tilde{\theta}}(1, 2) = \mathsf{d}_{\tilde{\theta}}(2, 1) = 80,$$

$$\mathsf{d}_{\tilde{\theta}}(1, 3) = \mathsf{d}_{\tilde{\theta}}(3, 1) = 1000,$$

$$\mathsf{d}_{\tilde{\theta}}(2, 3) = \mathsf{d}_{\tilde{\theta}}(3, 2) = 600.$$

We need to prove only (E_3), $\forall\, \zeta_1, \zeta_2, \zeta_3 \in \mathsf{G}$, note that

$$\begin{aligned}
80 &= \mathsf{d}_{\tilde{\theta}}(1,2) \leq \tilde{\theta}(1,2)[\mathsf{d}_{\tilde{\theta}}(1,3) + \mathsf{d}_{\tilde{\theta}}(3,2)] = (4)[1000 + 600] = (4)[1600] = 6400,\\
1000 &= \mathsf{d}_{\tilde{\theta}}(1,3) \leq \tilde{\theta}(1,3)[\mathsf{d}_{\tilde{\theta}}(1,2) + \mathsf{d}_{\tilde{\theta}}(2,3)]\\
&= (5)[80 + 600] = (5)[680] = 3400,\\
600 &= \mathsf{d}_{\tilde{\theta}}(2,3) \leq \tilde{\theta}(2,3)[\mathsf{d}_{\tilde{\theta}}(2,1) + \mathsf{d}_{\tilde{\theta}}(1,3)]\\
&= (6)[80 + 1000]\\
&= (6)[1080] = 6480.
\end{aligned}$$

Therefore,

$$\mathsf{d}_{\tilde{\theta}}(\zeta_1, \zeta_3) \leq \tilde{\theta}(\zeta_1, \zeta_3)[\mathsf{d}_{\tilde{\theta}}(\zeta_1, \zeta_2) + \mathsf{d}_{\tilde{\theta}}(\zeta_2, \zeta_3)].$$

$\Rightarrow (\mathsf{G}, \mathsf{d}_{\tilde{\theta}})$ is an extended b-MS but it is not a MS. Further, note that, taking $b = 7$, $(\mathsf{G}, \mathsf{d}_{\tilde{\theta}})$ becomes a b-MS.

Cauchy sequence and convergence concepts in extended b-MSs follow criteria similar to the MSs framework. Mlaiki et al. [37] introduced a novel extension of b-MSs, termed as controlled MSs. This extension incorporates a control function $\sigma(\zeta_1, \zeta_2)$ into the triangle inequality as follows:

Definition 2.5.7. Assume $\mathsf{G} \neq \emptyset$, and $\sigma : \mathsf{G} \times \mathsf{G} \to [1, \infty)$. Then a function $\mathsf{d}_\sigma : \mathsf{G} \times \mathsf{G} \to [0, \infty)$ is known as a controlled metric if $\forall\, \zeta_1, \zeta_2, \zeta_3 \in \mathsf{G}$, it satisfies the following properties:

(C_1): $\mathsf{d}_\sigma(\zeta_1, \zeta_2) = 0 \Leftrightarrow \zeta_1 = \zeta_2$;

(C_2): $\mathsf{d}_\sigma(\zeta_1, \zeta_2) = \mathsf{d}_\sigma(\zeta_2, \zeta_1)$;

(C_3): $\mathsf{d}_\sigma(\zeta_1, \zeta_3) \leq [\sigma(\zeta_1, \zeta_2)\mathsf{d}_\sigma(\zeta_1, \zeta_2) + \sigma(\zeta_2, \zeta_3)\mathsf{d}_\sigma(\zeta_2, \zeta_3)]$.

The pair $(\mathsf{G}, \mathsf{d}_\sigma)$ is said to be a controlled MS.

Remark 2.5.8.

(1): The class of extended b-MSs encompasses both b-MSs and MSs. Specifically, setting $\tilde{\theta}(\zeta_1, \zeta_3) = s$, $\mathsf{d}_{\tilde{\theta}}$ reduces to b-MSs, while setting $\tilde{\theta}(\zeta_1, \zeta_3) = 1$ yields a MSs. Consequently, every MS is a b-MS, and every b-MS is an extended b-MS, but the converses do not hold.

(2): If $\sigma(\zeta_1, \zeta_2) = s \geq 1 \; \forall \, \zeta_1, \zeta_2 \in \mathsf{G}$, then (G, d) is a b-MS. This implies that every b-MS is a controlled MS. However, the converse does not hold in general; specifically, a controlled MS may not be an extended b-MS when $\tilde{\theta} = \sigma$. The distinction can be further clarified through the following examples.

Example 2.5.9. Choose $\mathsf{G} = \{1, 2, \ldots\}$. Take $\mathsf{d} : \mathsf{G} \times \mathsf{G} \to [0, \infty)$ as

$$\mathsf{d}(\zeta_1, \zeta_2) = \begin{cases} 0, & \text{if } \zeta_1 = \zeta_2, \\ \frac{1}{\zeta_1}, & \text{if } \zeta_1 \text{ is even and } \zeta_2 \text{ is odd}, \\ 1, & \text{otherwise}, \\ \frac{1}{\zeta_2}, & \text{if } \zeta_1 \text{ is odd and } \zeta_2 \text{ is even}. \end{cases}$$

Consider $\sigma : \mathsf{G} \times \mathsf{G} \to [1, \infty)$ as

$$\sigma(\zeta_1, \zeta_2) = \begin{cases} \zeta_2, & \text{if } \zeta_1 \text{ is odd and } \zeta_2 \text{ is even}, \\ 1, & \text{otherwise}, \\ \zeta_1, & \text{if } \zeta_1 \text{ is even and } \zeta_2 \text{ is odd}. \end{cases}$$

We need to prove (C_3), for this consider.

Case 1: If $\mathsf{v} = \zeta_1$ or $\mathsf{v} = \zeta_2$, (C_3) is satisfied.

Case 2: If $\mathsf{v} \neq \zeta_1$ and $\mathsf{v} \neq \zeta_2$, (C_3) holds when $\zeta_1 = \zeta_2$. From this point onward, assume that $\zeta_1 \neq \zeta_2$. Consequently, we have $\zeta_1 \neq \zeta_2 \neq \mathsf{v}$.

It is also evident that condition (C_3) is satisfied in all of the following possible subcases:

(i): ζ_1 and v are even, while ζ_2 is odd.

(ii): ζ_1 is even, and both ζ_2 and v are odd.

(iii): ζ_1 and v are odd, while ζ_2 is even.

(iv): ζ_1 is odd, ζ_2 is odd, and v is even.

(v): All of ζ_1, ζ_2, and v are even.

(vi): ζ_1 and ζ_2 are even, and v is odd.

(vii): ζ_1 and ζ_2 are odd, and v is even.

$(viii)$: All of ζ_1, ζ_2, and v are odd.

$$\Rightarrow \mathsf{d}_\sigma \text{ is a controlled metric.}$$

Alternatively, for $\mathsf{n} = 2, 3, \ldots$, we obtain

$$\begin{aligned}\mathsf{d}_\sigma(2\mathsf{n}+1, 4\mathsf{n}+1) &= 1 \\ &> \frac{1}{\mathsf{n}} \\ &= \sigma(2\mathsf{n}+1, 4\mathsf{n}+1)\,[\mathsf{d}_\sigma(2\mathsf{n}+1, 2\mathsf{n}) + \mathsf{d}_\sigma(2\mathsf{n}, 4\mathsf{n}+1)].\end{aligned}$$

Thus, d_σ fails to be an extended b-metric when considered with the function $\sigma = \tilde{\theta}$.

Example 2.5.10. Take $\mathsf{G} = \{0, 1, 2\}$. Consider the function $\mathsf{d}_\sigma : \mathsf{G} \times \mathsf{G} \to \mathbb{R}^+$ defined as

$$\mathsf{d}_\sigma(0,0) = \mathsf{d}_\sigma(1,1) = \mathsf{d}_\sigma(2,2) = 0, \quad \mathsf{d}_\sigma(0,1) = \mathsf{d}_\sigma(1,0) = 1,$$

$$\mathsf{d}_\sigma(0,2) = \mathsf{d}_\sigma(2,0) = \frac{1}{2}, \quad \mathsf{d}_\sigma(1,2) = \mathsf{d}_\sigma(2,1) = \frac{2}{5}.$$

Let $\sigma : \mathsf{G} \times \mathsf{G} \to [1, \infty)$ be a symmetric function defined by

$$\sigma(0,0) = \sigma(1,1) = \sigma(2,2) = \sigma(0,2) = 1, \quad \sigma(1,2) = \frac{5}{4}, \quad \sigma(0,1) = \frac{11}{10}.$$

It is straightforward to verify that d_σ is a controlled metric. However, observe that

$$\mathsf{d}_\sigma(0,1) = 1 > \frac{99}{100} = \sigma(0,1)\,[\mathsf{d}_\sigma(0,2) + \mathsf{d}_\sigma(2,1)]\,.$$

This clearly demonstrates that d_σ does not satisfy the conditions of an extended b-metric with respect to the control function $\sigma = \tilde{\theta}$.

Cauchy sequence and convergence criteria in controlled MSs behave similarly to those in MSs.

Abdeljawad et al. [38] modify controlled MSs via two control functions $\sigma(\zeta_1, \zeta_2)$ and $\rho(\zeta_1, \zeta_2)$ by introducing the idea of double controlled MS as follows:

Definition 2.5.11. Assume $\mathsf{G} \neq \emptyset$, with non comparable functions $\sigma, \rho : \mathsf{G} \times \mathsf{G} \to [1, \infty)$. If $\mathsf{d}_{\sigma,\rho} : \mathsf{G} \times \mathsf{G} \to [0, \infty)$ satisfies the following properties:

(DC_1): $\mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2) = 0 \Leftrightarrow \zeta_1 = \zeta_2$;

(DC_2): $\mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2) = \mathsf{d}_{\sigma,\rho}(\zeta_2, \zeta_1)$;

(DC_3): $\mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_3) \leq [\sigma(\zeta_1, \zeta_2)\mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2) + \rho(\zeta_2, \zeta_3)\mathsf{d}_{\sigma,\rho}(\zeta_2, \zeta_3)]$,

$\forall\ \zeta_1, \zeta_2, \zeta_3 \in \mathsf{G}$. The $\mathsf{d}_{\sigma,\rho}$ is said to be a double controlled metric and the pair $(\mathsf{G}, \mathsf{d}_{\sigma,\rho})$ is called a double controlled MS.

Remark 2.5.12. A controlled MS is also a double controlled MS when taking the same function. However, the converse does not hold in general.

Example 2.5.13. Let $\mathsf{G} = [0, \infty)$. Define $\mathsf{d}_{\sigma,\rho} : \mathsf{G} \times \mathsf{G} \to [0, \infty)$ by

$$\mathsf{d}(\zeta_1, \zeta_2) = \begin{cases} 1, & \text{otherwise}, \\ \frac{1}{\zeta_1}, & \text{if } \zeta_1 \geq 1 \text{ and } \zeta_2 \in [0, 1), \\ \frac{1}{\zeta_2}, & \text{if } \zeta_2 \geq 1 \text{ and } \zeta_1 \in [0, 1), \\ 0, & \text{if } \zeta_1 = \zeta_2. \end{cases}$$

Consider $\sigma, \rho : \mathsf{G} \times \mathsf{G} \to [1, \infty)$ as

$$\rho(\zeta_1, \zeta_2) = \begin{cases} \max\{\zeta_1, \zeta_2\}, & \text{otherwise}, \\ 1, & \text{if } \zeta_1 < 1 \text{ and } \zeta_2 < 1, \end{cases}$$

and

$$\sigma(\zeta_1,\zeta_2)=\begin{cases}1, & \text{otherwise},\\ \zeta_1, & \text{if } \zeta_1,\zeta_2\geq 1,\end{cases}$$

The conditions (DC_1) and (DC_2) are satisfied. We assert that (DC_3) also holds.

(i): If $\zeta_3=\zeta_1$ or $\zeta_3=\zeta_2$ then (DC_3) holds. (ii): First, (DC_3) is verified in the case $\zeta_1=\zeta_2$. Now, consider $\zeta_1\neq\zeta_2$, which implies $\zeta_1\neq\zeta_2\neq\zeta_3$.

In the subcases ($\zeta_1\geq 1$ and $\zeta_2\in[0,1)$) and ($\zeta_2\geq 1$ and $\zeta_1\in[0,1)$), it is straightforward to see that (DC_3) holds.

Subcase 1: $\zeta_1,\zeta_2\geq 1$.

- If $\zeta_3\geq 1$, then (DC_3) is satisfied.
- If $\zeta_3\in[0,1)$, we have
$$1\leq\frac{1}{\zeta_1}+\zeta_2\cdot\frac{1}{\zeta_2},$$
which confirms that (DC_3) holds.

Subcase 2: $\zeta_1,\zeta_2<1$.

- If $\zeta_3\in[0,1)$, then (DC_3) clearly holds.
- If $\zeta_3\geq 1$, we have
$$1\leq\frac{1}{\zeta_3}+\zeta_3\cdot\frac{1}{\zeta_3},$$
which verifies (DC_3).

From the above cases, we conclude that $\mathsf{d}_{\sigma,\rho}$ defines a double controlled metric.

However, consider

$$\begin{aligned}\mathsf{d}_{\sigma,\rho}\Big(0,\frac{1}{2}\Big)&=1\\ &>\frac{2}{3}\\ &=\frac{1}{3}+\frac{1}{3}\\ &=\sigma(0,3)\,\mathsf{d}_{\sigma,\rho}(0,3)+\rho(3,\frac{1}{2})\,\mathsf{d}_{\sigma,\rho}(3,\frac{1}{2}).\end{aligned}$$

Hence, $\mathsf{d}_{\sigma,\rho}$ is not an extended b-metric when $\rho = \sigma$.

Example 2.5.14. Let $\mathsf{G} = \{0, 1, 2\}$. Consider d defined by

$\mathsf{d}(\zeta, \zeta_2)$	0	1	2
0	0	1	1
1	1	0	$\frac{2}{5}$
2	1	$\frac{2}{5}$	0

Define σ and ρ as

$\sigma(\zeta_1, \zeta_2) =$

	0	1	2
0	1	$\frac{11}{10}$	1
1	$\frac{11}{10}$	1	$\frac{5}{8}$
2	1	$\frac{5}{8}$	1

and

$\rho(\zeta_1, \zeta_2) =$

	0	1	2
0	1	$\frac{11}{10}$	$\frac{3}{2}$
1	$\frac{11}{10}$	$\frac{3}{2}$	1
2	$\frac{3}{2}$	$\frac{5}{4}$	1.

Then $(\mathsf{G}, \mathsf{d}_{\sigma,\rho})$ is a double controlled MS.

Double controlled MSs exhibit Cauchy sequence and convergence properties analogous to those in MSs.

The distance between two closed sets is a key concept in mathematics. It was first introduced by Pompeiu in 1905 and later developed by Hausdorff. This idea helps define how far apart two sets are and is used to study FPs in mathematics.

Definition 2.5.15. Assume that $\mathfrak{CB}(\mathsf{G})$ is the family of closed and bounded subsets of a MS (G, d). For $\zeta \in \mathsf{G}$ and $\mathsf{A}, \mathsf{B} \in \mathfrak{CB}(\mathsf{G})$, set

$$\mathsf{d}(\zeta, \mathsf{B}) = \inf_{\mathsf{b} \in \mathsf{B}} (\mathsf{d}(\zeta, \mathsf{b})).$$

Define $\mathrm{H} : \mathfrak{CB}(\mathsf{G}) \times \mathfrak{CB}(\mathsf{G}) \longrightarrow \mathbb{R}^+$ by

$$\mathrm{H}(\mathsf{A}, \mathsf{B}) = \max\left\{ \sup_{\mathsf{a} \in \mathsf{A}} \mathsf{d}(\mathsf{a}, \mathsf{B}), \sup_{\mathsf{b} \in \mathsf{B}} \mathsf{d}(\mathsf{b}, \mathsf{A}) \right\},$$

then $(\mathfrak{CB}(\mathsf{G}), \mathrm{H})$ is named as Pompeiu-Hausdorff MS. [75]

Example 2.5.16. Consider a MS (G, d), where d is a usual metric and $\mathsf{G} = \mathbb{R}$. Let $\mathsf{A} = [1, 23]$ and $\mathsf{B} = [25, 40]$ are subsets of G.

Here the Hausdorff distance between the set A and the set B is given as:

$$\mathrm{H}(\mathsf{A}, \mathsf{B}) = \max\{\sup_{a \in \mathsf{A}} \mathsf{d}(\mathsf{a}, \mathsf{A}), \sup_{\mathsf{b} \in \mathsf{B}} \mathsf{d}(\mathsf{A}, \mathsf{b})\},$$

we will find out $\mathsf{d}(\mathsf{a}, \mathsf{B}) = \inf_{\mathsf{b} \in \mathsf{B}}(\mathsf{d}(\mathsf{a}, \mathsf{b}))$ is corresponds to the smallest distance from a point $\mathsf{a} \in \mathsf{A}$ to the nearest point in B. Consider $\mathsf{a} = 13$. Then

$$\mathsf{d}(13, \mathsf{B}) = \inf_{25 \in \mathsf{B}}(\mathsf{d}(13, 25)) = |13 - 25| = 12.$$

Now

$$\mathsf{d}(\mathsf{A}, \mathsf{B}) = \mathsf{d}(1, 25) = \sup_{1 \in \mathsf{A}}\{\mathsf{d}(1, 25) : 25 \in \mathsf{B}\} = \mathsf{d}(1, 25) = |1 - 25| = 24.$$

Let $\mathsf{a} = 23$ and $\mathsf{b} = 40$ so

$\mathsf{d}(23, 40) = \mathsf{d}(23, 40) = |23 - 40| = 17.$

Thus,

$$\mathrm{H}(\mathsf{A}, \mathsf{B}) = \max\{24, 17\} = 24.$$

Example 2.5.17. Let $\mathsf{G} = \mathbb{R}$. Define $\mathsf{d} : \mathsf{G} \times \mathsf{G} \longrightarrow [0, \infty)$ by

$$\mathsf{d}(\zeta_1, \zeta_2) = |\zeta_1 - \zeta_2|.$$

Then (d, G) is a complete MS.

For $\mathsf{A} = [0, 20]$ and let $\mathsf{B} = [22, 31]$, we define the distance from a point $\mathsf{a} \in \mathsf{A}$ to the set B as

$$\mathsf{d}(\mathsf{a}, \mathsf{B}) = \inf_{\mathsf{b} \in \mathsf{B}} \mathsf{d}(\mathsf{a}, \mathsf{b}),$$

which represents the smallest distance from a to any point in B. Consider the case where $\mathsf{a} = 12$. Then,

$$\mathsf{d}(12, \mathsf{B}) = \inf_{22 \in \mathsf{B}} \mathsf{d}(12, 22) = 10.$$

It can be seen that for every $\mathsf{a} \in \mathsf{A}$, the point in B that attains the minimum distance is $\mathsf{b} = 22$. Hence,

$$\sup_{\mathsf{a} \in \mathsf{A}} \mathsf{d}(\mathsf{a}, \mathsf{B}) = \sup \left\{ \mathsf{d}(\mathsf{a}, 22) \mid \mathsf{a} \in \mathsf{A} \right\}.$$

Among all elements in A, the point $\mathsf{a} = 0$ gives the maximum distance to $\mathsf{b} = 22$. Therefore,

$$\sup \left\{ \mathsf{d}(\mathsf{a}, 22) \mid \mathsf{a} \in \mathsf{A} \right\} = \mathsf{d}(0, 22) = 22.$$

Similarly, we find that

$$\sup_{\mathsf{b} \in \mathsf{B}} \mathsf{d}(\mathsf{B}, \mathsf{A}) = \sup\{\mathsf{d}(\mathsf{b}, 20) \;\; \mathsf{b} \in \mathsf{B}\}.$$

The point $\mathsf{b} = 31$ in B maximizes this distance. Therefore

$$\sup\{\mathsf{d}(\mathsf{b}, 20) \;\; \mathsf{b} \in \mathsf{B}\} = \sup\{\mathsf{d}(31, 20) \;\; 20 \in \mathsf{B}\} = 11.$$

It follows that

$$\mathrm{H}(\mathsf{A}, \mathsf{B}) = \max \left\{ \sup_{\mathsf{a} \in \mathsf{A}} \mathsf{d}(\mathsf{a}, \mathsf{B}), \sup_{\mathsf{b} \in \mathsf{B}} \mathsf{d}(\mathsf{b}, \mathsf{A}) \right\} = 22$$

2.6 Banach Contraction Principle and its Generalization

The Banach FP theorem is a foundational result in the field of mathematical analysis, especially within functional analysis. It introduces a formal framework for understanding contraction mappings in MSs and offers a reliable method for establishing the convergence of iterative sequences to a FP. What makes the Banach FP result particularly valuable is its assurance of both existence and uniqueness of a

FP for any contraction mapping. This result is instrumental in solving equations, analyzing the behavior of dynamical systems, and constructing effective iterative numerical methods.

Definition 2.6.1. For a MS (G, d) the mapping $\Re_1 : \mathsf{G} \to \mathsf{G}$ is called a contraction on G if there is a positive real number $\mathsf{a} < 1$ such that $\forall\, \zeta_1, \zeta_2 \in \mathsf{G}$

$$\mathsf{d}(\Re_1\zeta_1, \Re_1\zeta_2) \leq \mathsf{a}\mathsf{d}(\zeta_1, \zeta_2) \quad (\mathsf{a} < 1).$$ [10]

Definition 2.6.2. For a MS (G, d) the mapping $\Re_1 \colon \mathsf{G} \to \mathsf{G}$ is called contractive mapping if for every $\zeta_1, \zeta_2 \in \mathsf{G}$:

$$\mathsf{d}(\Re_1\zeta_1, \Re_1\zeta_2) < \mathsf{d}(\zeta_1, \zeta_2), \quad \text{with} \quad \zeta_1 \neq \zeta_2.$$ [94]

Theorem 2.6.3. Consider a MS $\mathsf{G} = (\mathsf{G}, \mathsf{d})$, where $\mathsf{G} \neq \emptyset$. Suppose that G is complete and let $\Re_1 : \mathsf{G} \to \mathsf{G}$ be a contraction on G. Then $\Re_1$ has precisely one FP. [10]

Many authors have extended Banach contraction principle by using different contractive conditions. Some of the such results are stated below: The very first generalization of Banach contraction principle is given by Edelstein in which he consider compact space instead of complete space.

Theorem 2.6.4. Let G be a MS and $\Re_1$ be a contractive mapping of G into itself such that there exists a point $\zeta \in \mathsf{G}$ whose sequence of iterates $\{{\Re_1}^t\zeta\}$ contains a convergent sub sequence $\{{\Re_1}^{ti}\zeta\}$; then

$$\xi = \lim_{i\to\infty} {\Re_1}^{ti}\zeta \in \mathsf{G}$$

is a unique FP. [95]

The result given below is provided by Chatterjea in which a new type of contraction is applied for giving a new result.

Theorem 2.6.5. Let (G, d) be a complete MS. A self-mapping $\Re_1 : \mathsf{G} \to \mathsf{G}$ is a contractive mapping if $\exists\, \mathsf{k} \in [0, \frac{1}{2})$ such that

$$\mathsf{d}(\Re_1\zeta_1, \Re_1\zeta_2) \leq \mathsf{k}(\mathsf{d}(\zeta_2, \Re_1\zeta_1) + \mathsf{d}(\zeta_1, \Re_1\zeta_2)),\ \forall\, \zeta_1, \zeta_2 \in \mathsf{G}.$$

Then $\Re_1$ has a unique FP. [50]

Jleli and Samet [64] proposed a novel class of contractive maps and proved a corresponding FP theorem for these mappings within the framework of generalized MSs.

Definition 2.6.6. We denote by Θ the set of functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(Θ_1): θ is increasing;

(Θ_2): for each sequence $\{\mathsf{t}_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(\mathsf{t}_n) = 1 \Leftrightarrow \lim_{n\to\infty} \mathsf{t}_n = 0^+$;

(Θ_3): $\exists\, \mathsf{k} \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{\mathsf{t}\to 0^+} \dfrac{\theta(\mathsf{t}) - 1}{\mathsf{t}^{\mathsf{k}}} = \ell$.

Definition 2.6.7. Let $\theta \in \Theta$ and (G, d) be a MS. Then $\Re_1 : \mathsf{G} \to \mathsf{G}$ is known to be a Θ-contraction if $\exists\, 0 < \mathsf{k} < 1$ such that

$$\theta(\mathsf{d}(\Re_1\zeta_1, \Re_1\zeta_2)) \leq [\theta(\mathsf{d}(\zeta_1, \zeta_2))]^{\mathsf{k}}$$

for each $\zeta_1, \zeta_2 \in \mathsf{G}$ with $\mathsf{d}(\Re_1\zeta_1, \Re_1\zeta_2) > 0$. [64]

Using the concept of Θ-contractions, Jleli and Samet [64] introduced the subsequent result:

Theorem 2.6.8. Let (G, d) be a complete generalized MS and $\Re_1 : \mathsf{G} \to \mathsf{G}$ be a given map. Suppose that $\exists\, \theta \in \Theta$ and $\mathsf{k} \in (0, 1)$ such that $\forall\, \zeta_1, \zeta_2 \in \mathsf{G}$,

$$\mathsf{d}(\Re_1\zeta_1, \Re_1\zeta_2) > 0 \Rightarrow \theta(\mathsf{d}(\Re_1\zeta_1, \Re_1\zeta_2)) \leq [\theta(\mathsf{d}(\zeta_1, \zeta_2))]^{\mathsf{k}}.$$

Then $\Re_1$ has a unique FP.

Example 2.6.9. Define a mapping $\theta : (0, \infty) \longrightarrow (1, \infty)$ by

$$\theta(\mathsf{t}) = e^{\sqrt{\mathsf{t}e^{\mathsf{t}}}}.$$

Then one can easily verify θ satisfy condition (Θ_1) and (Θ_2) of Definition 2.6.6. To prove (Θ_3) consider

$$\begin{aligned} \lim_{\mathsf{t}\to 0^+} \frac{\theta(\mathsf{t}) - 1}{\mathsf{t}^{\mathsf{k}}} &= \lim_{\mathsf{t}\to 0^+} \frac{e^{\sqrt{\mathsf{t}e^{\mathsf{t}}}} - 1}{\mathsf{t}^{\mathsf{k}}} \\ &= \lim_{\mathsf{t}\to 0^+} \frac{e^{\sqrt{\mathsf{t}e^{\mathsf{t}}}}(e^{\mathsf{t}} + \mathsf{t}e^{\mathsf{t}})}{2\mathsf{k}\sqrt{\mathsf{t}e^{\mathsf{t}}} \times \mathsf{t}^{\mathsf{k}-1}}. \end{aligned}$$

Now $\forall\ \mathsf{k} \in (\frac{1}{2}, 1)$

$$\Rightarrow \lim_{\mathsf{t}\to 0^+} \frac{\theta(\mathsf{t}) - 1}{\mathsf{t}^{\mathsf{k}}} = \infty.$$

Some further examples of Θ-contraction are as follows:

Example 2.6.10. Let $\theta : (0, \infty) \longrightarrow (1, \infty)$ be defined by

(1): $\theta(\mathsf{t}) = e^{\sqrt{\mathsf{t}}}$.

(2): $\theta(\mathsf{t}) = 2 - \frac{2}{\pi} \arctan\left(\frac{1}{\mathsf{t}^{\alpha}}\right), \quad 0 < \alpha < 1, \quad \mathsf{t} > 0.$

Then $(1) - (2)$ satisfy all the properties of Θ.

Later on, motivated by this idea Ameer et al. [65] presented the following definition in the context of b-MS.

Definition 2.6.11. Let $s \geq 1$. We denote by Θ_s the set of all functions $\theta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, which fulfills the following conditions:

$(\Theta_1 s)$: θ is increasing;

$(\Theta_2 s)$: for each sequence $\{\mathsf{t}_n\} \subset \mathbb{R}^+$, $\lim_{n\to\infty} \theta(\mathsf{t}_n) = 1 \Leftrightarrow \lim_{n\to\infty} \mathsf{t}_n = 0^+$;

$(\Theta_3 s)$: $\exists\ \mathsf{k} \in (0, 1)$ and $l \in (0, \infty]$ so that $\lim_{\mathsf{t}\to 0^+} \frac{\theta(\mathsf{t}) - 1}{\mathsf{t}^{\mathsf{k}}} = l$;

$(\Theta_4 s)$: for each sequence $\{\mathsf{t}_n\} \subset \mathbb{R}^+$, such that $\theta(s\mathsf{t}_n) \leq [\theta(\mathsf{t}_{n-1})]^{\mathsf{k}}$ $\forall\, n \in \mathbb{N}$ then $\theta(s^n \mathsf{t}_n) \leq [\theta(s^{n-1}\mathsf{t}_{n-1})]^{\mathsf{k}}$ for some $\mathsf{k} \in (0,1)$ and $\forall\, n \in \mathbb{N}$.

Example 2.6.12. Define $\theta : (0,\infty) \to (1,\infty)$ by

$$\theta(\mathsf{t}) = e^{\sqrt{\mathsf{t}}}.$$

Then clearly, θ satisfies conditions $(\Theta_1 s)$-$(\Theta_4 s)$. Now we show only condition $(\Theta_4 s)$.

Suppose that, for some $\mathsf{k} \in (0,1)$ and $\forall\, n \in \mathbb{N}$, we have

$$e^{\sqrt{s^n \mathsf{t}_n}} \leq \left[\left(e^{\sqrt{\mathsf{t}_{n-1}}}\right)\right]^{\mathsf{k}}.$$

Then, we compute:

$$\begin{aligned} e^{\sqrt{s^n \mathsf{t}_n}} &= \left(e^{\sqrt{\mathsf{t}_n}}\right)^{\sqrt{s^n}} \\ &= \left[e^{\sqrt{s^n \mathsf{t}_n}}\right]^{\sqrt{s^{n-1}}} \\ &\leq \left[\left(e^{\sqrt{\mathsf{t}_{n-1}}}\right)^{\mathsf{k}}\right]^{\sqrt{s^{n-1}}} \\ &= \left(e^{\sqrt{s^{n-1}\mathsf{t}_{n-1}}}\right)^{\mathsf{k}}. \end{aligned}$$

Hence, $(\Theta_s 4)$ is true.

Note that also, $\theta(\mathsf{t}) = e^{\sqrt{\mathsf{t}}} \in \Theta_s$.

The idea of the $\mathtt{F}$-mappings and $\mathtt{F}$-contractions was presented by Wardowski [55] in 2012 as follows:

Definition 2.6.13. Let $\mathtt{F}\colon \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying the following conditions:

(F_1): $\forall\, \zeta_1, \zeta_2 \in \mathbb{R}^+$, such that $\zeta_1 < \zeta_2$, we have $\mathtt{F}(\zeta_1) < \mathtt{F}(\zeta_2)$.

(F_2): For each sequence $\{\zeta_{\mathsf{t}}\}_{\mathsf{t}\in\mathbb{N}}$ of positive numbers,

$$\lim_{\mathsf{t}\to\infty} \zeta_{\mathsf{t}} = 0 \quad \text{if and only if} \quad \lim_{\mathsf{t}\to\infty} \mathtt{F}(\zeta_{\mathsf{t}}) = -\infty.$$

(F_3): There is a real number $\mathsf{k} \in (0,1)$ such as

$$\lim_{\zeta \to 0^+} \zeta^{\mathsf{k}} \mathtt{F}(\zeta) = 0. \text{ [55]}$$

Throughout the dissertation $\nabla\mathtt{F}^*_\sigma$ denotes the class of functions $\mathtt{F} : \mathbb{R}^+ \longrightarrow \mathbb{R}$ that satisfy the conditions (F_1) to (F_3).

Definition 2.6.14. A mapping $\Re_1 : \mathsf{G} \to \mathsf{G}$ is said to be an $\mathtt{F}$-contraction if $\exists\ \tau > 0$ such that $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$,

$$\mathsf{d}(\Re_1\zeta_1, \Re_1\zeta_2) > 0 \Rightarrow \tau + \mathtt{F}(\mathsf{d}(\Re_1\zeta_1, \Re_1\zeta_2)) \leq \mathtt{F}(\mathsf{d}(\zeta_1, \zeta_2)), \tag{2.1}$$

where $\mathtt{F} \in \nabla\mathtt{F}^*_\sigma$. [55]

Example 2.6.15. Define a mapping $\mathtt{F} : \mathbb{R}^+ \to \mathbb{R}$ by

$$\mathtt{F}(\zeta) = \ln(\zeta) \text{ for } \quad \zeta > 0,$$

and let $\Re_1 : \mathsf{G} \to \mathsf{G}$ be defined as

$$\Re_1(\zeta) = \frac{\zeta}{2},$$

with d being the usual metric on $\mathbb{R}$. Then $\mathsf{d}(\Re_1\zeta_1, \Re_1\zeta_2) > 0$.
Now $\forall\ \zeta_1, \zeta_2 \in \mathbb{R}$, and $\mathtt{F}(\zeta) = \ln\zeta$ implies

$$\begin{aligned} \tau + \mathtt{F}(\mathsf{d}(\Re_1\zeta_1, \Re_1\zeta_2)) &= \tau + \ln\left(\left|\frac{\zeta_1}{2} - \frac{\zeta_2}{2}\right|\right) \\ &= \tau + \ln|\zeta_1 - \zeta_2| - \ln 2 \\ &= \tau - .69314 + \frac{1}{2}|\zeta_1 - \zeta_2|. \end{aligned}$$

By taking $0 < \tau \leq .69314$ we get

$$\tau + \mathtt{F}(\mathsf{d}(\Re_1\zeta_1, \Re_1\zeta_2)) \leq \mathtt{F}(\mathsf{d}(\zeta_1, \zeta_2)).$$

Hence, $\Re_1$ is a $\mathtt{F}$-contraction.

Remark 2.6.16. From conditions $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$, it is easy to conclude that every $\mathtt{F}$-contraction $\Re_1$ is a contractive mapping, i.e.,

$$\mathsf{d}(\Re_1\zeta_1, \Re_1\zeta_2) < \mathsf{d}(\zeta_1, \zeta_2)$$

$\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$, with $\Re_1\zeta_1 \neq \Re_1\zeta_2$. Thus, every $\mathtt{F}$-contraction is a continuous mapping. [55].

The famous Wardowski [55] result proved on $\mathtt{F}$-mapping by using $\mathtt{F}$-contraction is presented below:

Theorem 2.6.17. Let (G, d) be a complete MS and suppose that $\Re_1 : \mathsf{G} \to \mathsf{G}$ be a $\mathtt{F}$-contraction. Then $\Re_1$ has a unique FP $\zeta^* \in \mathsf{G}$ and for every $\zeta_0 \in \mathsf{G}$ a sequence $\{{\Re_1}^n\zeta_0\}_{n\in N}$ is convergent to ζ^*. [55]

Example 2.6.18. The following mappings $\mathtt{F}\colon \mathbb{R}^+ \to \mathbb{R}$ are examples of $\mathtt{F}$-mappings:

(i)

$$\mathtt{F}(\zeta) = -\frac{1}{\sqrt{\zeta}} \quad \text{and} \quad \zeta > 0.$$

(ii)

$$\mathtt{F}(\zeta) = \ln\zeta + \zeta \quad \text{and} \quad \zeta > 0.$$

Definition 2.6.19. Consider the mappings $\Re_1\colon \mathsf{G} \to \mathsf{G}$ and $\alpha\colon \mathsf{G}\times\mathsf{G} \to [0,\infty)$. $\Re_1$ is an α-admissible if $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$, we have

$$\alpha(\zeta_1, \zeta_2) \geq 1 \Rightarrow \alpha(\Re_1\zeta_1, \Re_1\zeta_2) \geq 1. \text{ [96]}$$

Example 2.6.20. Let $\mathsf{G} = [0,\infty)$. Define a function $\Re_1 : \mathsf{G} \to \mathsf{G}$ by

$$\Re_1(\zeta) = \begin{cases} \ln(\zeta), & \text{if } \zeta \neq 0, \\ 3, & \text{if } \zeta = 0, \end{cases}$$

and define a function $\alpha : \mathsf{G} \times \mathsf{G} \to [0, \infty)$ by

$$\alpha(\zeta_1, \zeta_2) = \begin{cases} 3, & \text{if } \zeta_1 \geq \zeta_2, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\Re_1$ is α-admissible.

After the introduction of $\mathtt{F}$-contraction by Wardowski [55] in 2012, different authors established many interesting results in this setting. In this perspective Secelean [97], Piri et al. [98], Consentino et al. [99], Hussain et al. [100] and Sgroi et al. [56] used $\mathtt{F}$-contraction for different contraction conditions. Some more extensions can be seen in literature, for examples, [101, 102].

Chapter 3

FP Results via FMs in b-MSs

In this chapter we propose a new type of contractive condition, referred to as the (P, ψ)-type almost contraction, tailored for FMs within the setting of b-MSs. Using this framework, we derive several FP results for FMs in complete b-MSs. To illustrate the applicability of the main theorem, a concrete example is constructed that fulfills all the necessary conditions.

Moreover, we apply the developed FP theorems to demonstrate the existence of solutions for a second-order nonlinear boundary value problem by converting it into an equivalent FP formulation. Several corollaries are also derived from the principal results, further emphasizing the broad scope and applicability of the proposed framework. These contributions not only broaden but also unify various established FP theorems found in existing literature.

3.1 Chapter Layout

The following structure is adopted in this chapter to enhance understanding:

(i): The chapter begins with some basic definitions that are helpful for comprehending the subsequent content.

(ii): In Section 3.2, motivated by the concept of (P, ψ)-type almost contractive conditions in MS, we introduce the (P, ψ)-type almost contractive condition for FMs in b-MS and establish some FP theorems. Several corollaries are derived from the main results, and an example is provided to validate our findings.

(iii): In Section 3.3, MMs are utilized to further support our results. Various corollaries are formulated based on different mappings and contraction conditions.

(iv): In Section 3.4, to substantiate our main results, we apply them to establish the existence of a solution to a second-order nonlinear boundary value problem.

(v): In Section 3.5, some concluding remarks are presented to facilitate a better understanding of the chapter.

Definition 3.1.1. A fuzzy subset F of G is an approximate quantity iff its α-level set is a compact convex subset (non fuzzy) of G for each $\forall \ \in [0,1]$, and

$$\sup_{\zeta\in\mathsf{G}} \mathrm{F}(\zeta) = 1. \ [74]$$

Next, we introduce a notion of distance between approximate quantities. Throughout this work, approximate quantities are represented by elements of $\mathfrak{D}(\mathsf{G})$. Let $\mathsf{A}, \mathsf{B} \in \mathfrak{D}(\mathsf{G})$ for $\alpha \in [0,1]$, and consider

$$\begin{aligned} \mathrm{Q}_\alpha(\mathsf{A},\mathsf{B}) &= \inf_{\zeta_1\in[\mathsf{A}]_\alpha,\zeta_2\in[\mathsf{B}]_\alpha} (\mathsf{d}_b(\zeta_1,\zeta_2)), \\ \mathcal{D}_\alpha(\mathsf{A},\mathsf{B}) &= \mathrm{H}([\mathsf{A}]_\alpha,[\mathsf{B}]_\alpha) \\ \mathcal{D}(\mathsf{A},\mathsf{B}) &= \sup_\alpha \mathcal{D}_\alpha(\mathsf{A},\mathsf{B}). \end{aligned}$$

The mapping Q_α is called an α-space, $\mathcal{D}_\alpha$ is called an α-distance, and $\mathcal{D}$ denotes the distance between A and B.

Definition 3.1.2. A FS F in a b-metric linear space is called an approximate quantity iff F_α is compact and convex in G $\forall \ \alpha \in [0,1]$ and

$$\sup_{\zeta\in\mathsf{G}} \mathrm{F}(\zeta) = 1.$$

Lemma 3.1.3. Let $\mathsf{G} \neq \emptyset$. A point $\zeta^* \in \mathsf{G}$ is a FP of the FM $\mathcal{F}(\mathsf{G})$ iff ζ^* is a FP of set-valued mapping $\hat{\Re}_1 : \mathsf{G} \longrightarrow \mathfrak{CB}(\mathsf{G})$. [92]

Lemma 3.1.4. Every sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ of elements from a b-MS $(\mathsf{G}, \mathsf{d}_b)$ satisfying the property

$$\mathsf{d}_b(\zeta_{n+1}, \zeta_n) \leq K\mathsf{d}_b(\zeta_n, \zeta_{n-1}), \quad \forall\, K \in [0, 1),$$

is a Cauchy sequence. [86, Lemma 2.1]

Lemma 3.1.5. Assume $(\mathsf{G}, \mathsf{d}_b)$ being a complete b-MS. For $\mathsf{A}, \mathsf{B} \in \mathfrak{CB}(\mathsf{G})$ and $\mathrm{v} \in \mathsf{A}$

$$\mathsf{d}(\mathrm{v}, \mathsf{B}) \leq \mathrm{H}(\mathsf{A}, \mathsf{B}). \text{ [86, } \textit{Lemma } 1.3]$$

Lemma 3.1.6. Assume $\mathfrak{d}$ be a metric linear space. If $\Re_1 : \mathfrak{d} \longrightarrow \mathfrak{D}(\mathfrak{d})$ be a FM, $\mathsf{A}, \mathsf{B} \in \mathfrak{CB}(\mathsf{G})$ and $\mathrm{v}_0 \in \mathfrak{d}$ then there exist $\mathrm{v} \in \mathcal{V}$ so that $\{\mathrm{v}\} \subset \Re_1(\mathrm{v}_0)$

$$\mathsf{d}(\mathrm{v}, \mathsf{B}) \leq \mathrm{H}(\mathsf{A}, \mathsf{B}).$$

Lemma 3.1.7. Assume $(\mathsf{G}, \mathsf{d}_b)$ being a complete b-MS. Let $\mathsf{A}, \mathsf{B} \in \mathfrak{CB}(\mathsf{G})$ and $\mathrm{v} \in \mathsf{A}$ then for any $\mathrm{w} \in \mathsf{B}$,

$$\mathsf{d}_b(\mathrm{v}, \mathsf{B}) \leq \mathsf{d}_b(\mathrm{v}, \mathrm{w}). \text{ [86, } \textit{Lemma } 1.3]$$

Lemma 3.1.8. Assume $(\mathsf{G}, \mathsf{d}_b)$ be a complete b-MS. If there exist some $\zeta^* \in \mathsf{G}$ and $\Re_1, \Re_2 : \mathsf{G} \longrightarrow \mathcal{F}(\mathsf{G})$ be the FMs such that $\Re_1(\zeta)$ is a nonempty compact set for each $\zeta \in \mathsf{G}$.

Then $\zeta^* \in \Re_1(\zeta^*)$ if and only if $\Re_1(\zeta^*)(\zeta^*) \geq \Re_1(\zeta^*)(\zeta)\ \forall\ \zeta \in \mathsf{G}$.

Definition 3.1.9. Let $(\mathsf{G}, \mathsf{d}_b)$ be a complete b-MS and $\Re_1, \Re_2$ be the pair of FMs. Let $\alpha_{\Re_1(\zeta)}$ and $\alpha_{\Re_2(\zeta)} \in (0, 1]$. Then, $\zeta \in \mathsf{G}$ is said to be a common $\alpha-$fuzzy FP if $\zeta \in [\Re_1(\zeta)]_{\alpha_{\Re_1(\zeta)}} \cap [\Re_2(\zeta)]_{\alpha_{\Re_2(\zeta)}}$,

where $[\Re_1(\zeta)]_{\alpha_{\Re_1(\zeta)}}$ and $[\Re_2(\zeta)]_{\alpha_{\Re_2(\zeta)}}$ are the $\alpha-$level sets of FSs $\Re_1(\zeta)$ and $\Re_2(\zeta)$ respectively. [103]

3.2 (P, ψ) Type Almost Contractive Condition in b-MSs.

Throughout the chapter, d_b is considered as continuous, complete b-MS, and $P \in \upsilon$ is continuous. Motivated by the idea of (P,ψ) type almost contractive condition in MS we will introduced (P,ψ) type almost contractive condition for FMs in b-MS as follows:

Definition 3.2.1. Assume $(\mathsf{G},\mathsf{d}_b)$ is a b-MS with $s \geq 1$, then the pair of FM $\Re_1, \Re_2 : \mathsf{G} \to \mathcal{F}(\mathsf{G})$ is said to be (P,ψ) type almost contractive mapping. If $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$, $P \in \upsilon$, $\psi \in \kappa$ and $\mathfrak{L} \geq 0$,
$\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) > 0$,

$$\Rightarrow \psi(\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) \leq P(\psi(\mathtt{M}(\zeta_1,\zeta_2))) + \mathfrak{L}\Xi(\zeta_1,\zeta_2),$$

with

$$\begin{aligned}&\mathtt{M}(\zeta_1,\zeta_2)\\&= \frac{1}{s}\max\Bigg\{\mathsf{d}_b(\zeta_1,[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}), \mathsf{d}_b(\zeta_1,\zeta_2), \mathsf{d}_b(\zeta_2,[\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}),\\&\qquad \frac{\{\mathsf{d}_b(\zeta_2,[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}} + \mathsf{d}_b(\zeta_1,[\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}))\}}{2s}\Bigg\},\end{aligned}$$

and

$$\begin{aligned}\Xi(\zeta_1,\zeta_2) = \min\Big\{&\mathsf{d}_b(\zeta_1,[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}), \mathsf{d}_b(\zeta_2,[\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}),\\&\mathsf{d}_b(\zeta_1,[\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}), \mathsf{d}_b(\zeta_2,[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}})\Big\},\end{aligned}$$

here, $[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}$, $[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}} \in \mathfrak{CB}(\mathsf{G})$ are non empty and $\alpha_{\Re_1(\zeta_1)}$, $\alpha_{\Re_2(\zeta_1)} \in (0,1]$.

Theorem 3.2.2. Assume $(\mathsf{G},\mathsf{d}_b)$ be a complete b-MS and let $\Re_1, \Re_2$ be the pair of (P,ψ) type almost contractive mapping. Then, $\exists$ h $\in \mathsf{G}$ such that $\mathrm{h} \in \{[\Re_1(\mathrm{h})]_{\alpha_{\Re_1(\mathrm{h})}} \cap [\Re_2(\mathrm{h})]_{\alpha_{\Re_2(\mathrm{h})}}\}$.

Proof. Let $\zeta_0 \in \mathsf{G}$. By hypothesis there exist $\alpha_{\Re_1(\zeta_0)} \in (0,1]$ such that $[\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}$ is a non empty closed and bounded subset of G.

Let $\zeta_1 \in [\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}$ and $\exists\ \alpha_{\Re_2(\zeta_1)} \in (0,1]$

such that $[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}$ is a non empty and $\in \mathfrak{CB}(\mathsf{G})$.

Since ψ is non decreasing and by using Lemma 3.1.5 and Definition 3.2.1.

$$\begin{aligned}&\psi(\mathsf{d}_b(\zeta_1, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}))\\ &\leq \psi(\mathrm{H}([\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}})) \leq P(\psi(\mathtt{M}(\zeta_0,\zeta_1))) + \mathfrak{L}\Xi(\zeta_0,\zeta_1),\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}&\mathtt{M}(\zeta_0,\zeta_1)\\ &= \frac{1}{s}\max\Big\{\mathsf{d}_b(\zeta_0,\zeta_1), \mathsf{d}_b(\zeta_0, [\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}), \mathsf{d}_b(\zeta_1, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}),\\ &\frac{\{\mathsf{d}_b(\zeta_0, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}) + \mathsf{d}_b(\zeta_1, [\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}})\}}{2s}\Big\},\end{aligned}$$

and

$$\begin{aligned}\Xi(\zeta_0,\zeta_1) = \min\Big\{&\mathsf{d}_b(\zeta_0, [\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}), \mathsf{d}_b(\zeta_1, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}),\\ &\mathsf{d}_b(\zeta_0, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}), \mathsf{d}_b(\zeta_1, [\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}})\Big\}.\end{aligned}$$

As ψ is continuous,

$$\Rightarrow \psi(\mathsf{d}_b(\zeta_1, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}})) = \inf_{\mathrm{h}\in[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}} \psi(\mathsf{d}_b(\mathrm{h},\zeta_1)).$$

Thus $\exists\ \zeta_2 \in [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}$ such that,

$$\psi(\mathsf{d}_b(\zeta_1, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}})) = \psi(\mathsf{d}_b(\zeta_1,\zeta_2)).$$

So (3.1) becomes,

$$\psi(\mathsf{d}_b(\zeta_1,\zeta_2)) \leq P(\psi(\mathtt{M}(\zeta_0,\zeta_1))) + \mathfrak{L}\Xi(\zeta_0,\zeta_1). \tag{3.2}$$

Now,

$$\begin{aligned}
\mathtt{M}(\zeta_0, \zeta_1) = \\
&\frac{1}{s}\max\Big\{\mathsf{d}_b(\zeta_0,\zeta_1), \mathsf{d}_b(\zeta_0,\zeta_1), \mathsf{d}_b(\zeta_1,\zeta_2), \\
&\frac{\{\mathsf{d}_b(\zeta_0,\zeta_2) + \mathsf{d}_b(\zeta_1,\zeta_1)\}}{2s}\Big\}.
\end{aligned}$$

$$\mathtt{M}(\zeta_0,\zeta_1) \leq \frac{1}{s}\max\Big\{\mathsf{d}_b(\zeta_0,\zeta_1), \mathsf{d}_b(\zeta_1,\zeta_2)\Big\}.$$

$$\begin{aligned}
\Xi(\zeta_0,\zeta_1) &= \min\Big\{\mathsf{d}_b(\zeta_0,\zeta_1), \mathsf{d}_b(\zeta_1,\zeta_2), \mathsf{d}_b(\zeta_0,\zeta_2), \mathsf{d}_b(\zeta_1,\zeta_1)\Big\}, \\
&= \mathsf{d}_b(\zeta_1,\zeta_1) \\
&= 0.
\end{aligned}$$

If we take,

$$\max\Big\{\mathsf{d}_b(\zeta_0,\zeta_1), \mathsf{d}_b(\zeta_1,\zeta_2)\Big\} = \mathsf{d}_b(\zeta_1,\zeta_2).$$

Then (3.2) becomes,

$$\begin{aligned}
\psi(\mathsf{d}_b(\zeta_1,\zeta_2)) &\leq P(\psi(\mathtt{M}(\zeta_1,\zeta_2))) \leq P(\psi(\frac{1}{s}\mathsf{d}_b(\zeta_1,\zeta_2))) < \psi(\frac{1}{s}\mathsf{d}_b(\zeta_1,\zeta_2)), \\
\Rightarrow \psi(\mathsf{d}_b(\zeta_1,\zeta_2)) &\leq \psi(\frac{1}{s}\mathsf{d}_b(\zeta_1,\zeta_2)) \\
&< \psi(\mathsf{d}_b(\zeta_1,\zeta_2)),
\end{aligned}$$

a contradiction. Hence

$$\max\Big\{\mathsf{d}_b(\zeta_0,\zeta_1), \mathsf{d}_b(\zeta_1,\zeta_2)\Big\} = \mathsf{d}_b(\zeta_0,\zeta_1).$$

So (3.2) becomes,

$$\begin{aligned}
\psi(\mathsf{d}_b(\zeta_1,\zeta_2)) &\leq P(\psi(\mathtt{M}(\zeta_0,\zeta_1))) \\
&\leq P(\psi(\frac{1}{s}\mathsf{d}_b(\zeta_0,\zeta_1))), \\
\Rightarrow \psi(\mathsf{d}_b(\zeta_1,\zeta_2)) &\leq P(\psi(\frac{1}{s}\mathsf{d}_b(\zeta_0,\zeta_1))) \qquad (3.3) \\
&\leq P(\psi(\mathsf{d}_b(\zeta_0,\zeta_1))) \\
&< \psi(\mathsf{d}_b(\zeta_0,\zeta_1)).
\end{aligned}$$

Now there exist $\alpha_{\Re_1(\zeta_2)} \in (0,1]$ such that $[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}$ is a non empty $\in \mathfrak{CB}(\mathsf{G})$. Since ψ is non decreasing along with Lemma 3.1.5, (3.3) becomes,

$$\begin{aligned} &\psi(\mathsf{d}_b([\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}, \zeta_2)) \\ &\leq \psi(\mathrm{H}([\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}})) \leq P(\psi(\mathtt{M}(\zeta_1,\zeta_2))) + \mathfrak{L}\Xi(\zeta_1,\zeta_2), \end{aligned} \tag{3.4}$$

with

$$\begin{aligned} &\mathtt{M}(\zeta_1,\zeta_2) \\ &= \frac{1}{s}\max\Bigg\{\mathsf{d}_b(\zeta_2, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}), \mathsf{d}_b(\zeta_1,\zeta_2), \mathsf{d}_b(\zeta_1, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}), \\ &\frac{\{\mathsf{d}_b(\zeta_2, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}) + \mathsf{d}_b(\zeta_1, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}})\}}{2s}\Bigg\}, \end{aligned}$$

$$\begin{aligned} \Xi(\zeta_1,\zeta_2) = \min\Big\{&\mathsf{d}_b(\zeta_2, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}), \mathsf{d}_b(\zeta_1, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}), \\ &\mathsf{d}_b(\zeta_2, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}), \mathsf{d}_b(\zeta_1, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}})\Big\}. \end{aligned}$$

As ψ is continuous,

$$\Rightarrow \psi(\mathsf{d}_b(\zeta_2, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}})) = \inf_{\mathrm{u}_1 \in [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}} \psi(\mathsf{d}_b(\zeta_2, \mathrm{u}_1)).$$

Thus there exist $\zeta_3 \in [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}$ such that,

$$\psi(\mathsf{d}_b(\zeta_2, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}) = \psi(\mathsf{d}_b(\zeta_2,\zeta_3)).$$

So (3.4) becomes,

$$\psi(\mathsf{d}_b(\zeta_2,\zeta_3)) \leq P(\psi(\mathtt{M}(\zeta_1,\zeta_2))) + \mathfrak{L}\Xi(\zeta_1,\zeta_2). \tag{3.5}$$

Now

$$\begin{aligned} \mathtt{M}(\zeta_1,\zeta_2) \leq &\frac{1}{s}\max\left\{\mathsf{d}_b(\zeta_1,\zeta_2), \mathsf{d}_b(\zeta_2,\zeta_3), \mathsf{d}_b(\zeta_1,\zeta_2), \frac{\{\mathsf{d}_b(\zeta_1,\zeta_3) + \mathsf{d}_b(\zeta_2,\zeta_2)\}}{2s}\right\}, \\ = &\frac{1}{s}\max\Big\{\mathsf{d}_b(\zeta_1,\zeta_2), \mathsf{d}_b(\zeta_2,\zeta_3)\Big\}. \end{aligned}$$

$$\begin{aligned}\Xi(\zeta_1,\zeta_2) =& \min\left\{\mathsf{d}_b(\zeta_2,\zeta_3),\mathsf{d}_b(\zeta_1,\zeta_2),\mathsf{d}_b(\zeta_2,\zeta_2),\mathsf{d}_b(\zeta_1,\zeta_3)\right\},\\ =&\mathsf{d}_b(\zeta_2,\zeta_2)\\ =&0.\end{aligned}$$

If we take,

$$\max\left\{\mathsf{d}_b(\zeta_1,\zeta_2),\mathsf{d}_b(\zeta_2,\zeta_3)\right\} = \mathsf{d}_b(\zeta_2,\zeta_3).$$

Then (3.5) becomes,

$$\begin{aligned}\psi(\mathsf{d}_b(\zeta_2,\zeta_3)) &\le P(\psi(\mathtt{M}(\zeta_1,\zeta_2)))\\ &\le P(\psi(\frac{1}{s}\mathsf{d}_b(\zeta_2,\zeta_3)))\\ &< \psi(\frac{1}{s}\mathsf{d}_b(\zeta_2,\zeta_3)),\end{aligned}$$

$$\begin{aligned}\Rightarrow \psi(\mathsf{d}_b(\zeta_2,\zeta_3)) &\le \psi(\frac{1}{s}\mathsf{d}_b(\zeta_2,\zeta_3))\\ &< \psi(\mathsf{d}_b(\zeta_2,\zeta_3)),\end{aligned} \tag{3.6}$$

a contradiction. Thus

$$\max\left\{\mathsf{d}_b(\zeta_1,\zeta_2),\mathsf{d}_b(\zeta_2,\zeta_3)\right\} = \mathsf{d}_b(\zeta_1,\zeta_2).$$

So (3.5) becomes,

$$\psi(\mathsf{d}_b(\zeta_2,\zeta_3)) \le P(\psi(\frac{1}{s}\mathsf{d}_b(\zeta_1,\zeta_2))) \le P(\psi(\mathsf{d}_b(\zeta_1,\zeta_2))) < \psi(\mathsf{d}_b(\zeta_1,\zeta_2)). \tag{3.7}$$

Using this procedure, we generate a sequence

$\{\zeta_n\} \in \mathsf{G} \;\; \forall\, n \in \mathbb{N}$

such that

$\zeta_{2n+2} \in [\Re_2(\zeta_{2n+1})]_{\alpha_{\Re_2(\zeta_{2n+1})}}$ and $\zeta_{2n+1} \in [\Re_1(\zeta_{2n})]_{\alpha_{\Re_1(\zeta_{2n})}}$ such that

$$\begin{aligned}\psi(\mathsf{d}_b(\zeta_{2n+1},\zeta_{2n+2})) &\le P(\psi(\frac{1}{s}\mathsf{d}_b(\zeta_{2n},\zeta_{2n+1})))\\ &< \psi(\frac{1}{s}\mathsf{d}_b(\zeta_{2n},\zeta_{2n+1})), \qquad \forall\, n \in \mathbb{N}.\end{aligned} \tag{3.8}$$

Similarly we conclude that

$$\psi(\mathsf{d}_b(\zeta_{2n+2},\zeta_{2n+3})) < \psi(\frac{1}{s}\mathsf{d}_b(\zeta_{2n},\zeta_{2n+1})) \\ < \psi(\mathsf{d}_b(\zeta_{2n},\zeta_{2n+1})). \tag{3.9}$$

Thus from (3.8) and (3.9) we obtain,

$$\psi(\mathsf{d}_b(\zeta_n,\zeta_{n+1})) \leq P(\psi(\frac{1}{s}\mathsf{d}_b(\zeta_{n-1},\zeta_n))) < (\psi(\frac{1}{s}\mathsf{d}_b(\zeta_{n-1},\zeta_n))).$$

Since ψ is non decreasing

$$\mathsf{d}_b(\zeta_n,\zeta_{n+1}) < \frac{1}{s}(\mathsf{d}_b(\zeta_{n-1},\zeta_n)). \tag{3.10}$$

This implies that,

$$\psi(\mathsf{d}_b(\zeta_n,\zeta_{n+1})) \leq P(\psi(\mathsf{d}_b(\zeta_{n-1},\zeta_n))) \\ \leq P^2(\psi(\mathsf{d}_b(\zeta_{n-2},\zeta_{n-1}))). \\ \leq \cdots \leq P^n(\psi(\mathsf{d}_b(\zeta_0,\zeta_1))).$$

Since P is a comparison function, it leads to

$$0 \leq \lim_{n\longrightarrow\infty} P(\psi(\mathsf{d}_b(\zeta_n,\zeta_{n+1}))) \leq \lim_{n\longrightarrow\infty} P^n(\psi(\mathsf{d}_b(\zeta_0,\zeta_1))) = 0.$$

It gives us,

$$\lim_{n\longrightarrow\infty} P(\psi(\mathsf{d}_b(\zeta_n,\zeta_{n+1})) = 0.$$

$$\Rightarrow \lim_{n\longrightarrow\infty} \mathsf{d}_b(\zeta_n,\zeta_{n+1}) = 0. \tag{3.11}$$

We aim to prove that $\{\zeta_n\} \in \mathsf{G}$ is a Cauchy sequence.

From (3.10) with Lemma 3.1.4 it is clear that $\{\zeta_n\} \in \mathsf{G}$ is a Cauchy sequence. Since G is complete b-MS, $\{\zeta_n\}$ converges to $\mathrm{h}^* \in \mathsf{G}$, that is,

$$\lim_{\mathrm{n}\longrightarrow\infty} \mathsf{d}_b(\zeta_n,\mathrm{h}^*) = 0.$$

We claim that $\{\mathrm{h}^*\} \in [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}}$.

Assume on contrary that h^* does not belong to $[\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}}$ (that is, $\mathsf{d}_b(\mathrm{h}^*, [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}}) > 0$),

Then there are $n_0 \in \mathbb{N}$ and a sequence $\{\zeta_{n_k}\}$ of $\mathrm{g_n}$

so that

$\mathsf{d}_b(\zeta_{2n_k+1}, [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}}) > 0$, for all $n_k \geq n_0$. Since $\mathsf{d}_b(\zeta_{2n_k+1}, [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}}) > 0$, so by (υ_1), we have

$$\begin{aligned} \psi(\mathsf{d}_b(\zeta_{2n_k+1}, [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}})) &\leq \psi(\mathrm{H}([\Re_1(\zeta_{2n_k})]_{\alpha_{\Re_1(\zeta_{2n_k})}}, [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}})) \\ &\leq P(\psi(\mathtt{M}(\zeta_{2n_k}, \mathrm{h}^*))) + \mathfrak{L}\Xi(\zeta_{2n_k}, \mathrm{h}^*), \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} &\mathtt{M}(\zeta_{2n_k}, \mathrm{h}^*) \\ &= \frac{1}{s} \max\left\{ \mathsf{d}_b(\zeta_{2n_k}, \mathrm{h}^*), \mathsf{d}_b(\zeta_{2n_k}, [\Re_1(\zeta_{2n_k})]_{\alpha_{\Re_1(\zeta_{2n_k})}}), \mathsf{d}_b(\mathrm{h}^*, [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}}), \right. \\ &\left. \frac{\{\mathsf{d}_b(\zeta_{2n_k}, [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}}) + \mathsf{d}_b(\mathrm{h}^*, [\Re_1(\zeta_{2n_k})]_{\alpha_{\Re_1(\zeta_{2n_k})}})\}}{2s} \right\}. \end{aligned}$$

$$\begin{aligned} &\mathtt{M}(\zeta_{2n_k}, \mathrm{h}^*) \\ &\leq \frac{1}{s} \max\left\{ \mathsf{d}_b(\zeta_{2n_k}, \mathrm{h}^*), \mathsf{d}_b(\zeta_{2n_k}, \zeta_{2n_k+1}), \mathsf{d}_b(\mathrm{h}^*, [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2}(\mathrm{h}^*)}), \right. \\ &\left. \frac{\{\mathsf{d}_b(\zeta_{2n_k}, [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}}) + \mathsf{d}_b(\mathrm{h}^*, \zeta_{2n_k+1})\}}{2s} \right\}. \end{aligned}$$

$$\begin{aligned} &\Xi(\zeta_{2n_k}, \mathrm{h}^*) \\ &= \min\left\{ \mathsf{d}_b(\zeta_{2n_k}, [\Re_1(\zeta_{2n_k})]_{\alpha_{\Re_1(\zeta_{2n_k})}}), \mathsf{d}_b(\mathrm{h}^*, [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}}), \right. \\ &\left. \mathsf{d}_b(\zeta_{2n_k}, [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}}) + \mathsf{d}_b(\mathrm{h}^*, [\Re_1(\zeta_{2n_k})]_{\alpha_{\Re_1(\zeta_{2n_k})}}) \right\}. \\ &\leq \min\left\{ \mathsf{d}_b(\zeta_{2n_k}, \zeta_{2n_k+1}), \mathsf{d}_b(\mathrm{h}^*, [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}}), \right. \\ &\left. \mathsf{d}_b(\zeta_{2n_k}, [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2}(\mathrm{h}^*)}) + \mathsf{d}_b(\mathrm{h}^*, \zeta_{2n_k+1}) \right\}. \end{aligned}$$

By taking the limit as n approaches to infinity and using the continuity of ψ and P, equation (3.12) becomes,

$$\begin{aligned}\psi(\mathsf{d}_b(\mathrm{h}^*, [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}})) &\leq P(\psi(\mathsf{d}_b(\mathrm{h}^*, [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}}))) + 0,\\ &< (\psi(\mathsf{d}_b(\mathrm{h}^*, [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}}))).\end{aligned}$$

Which is again a contradiction.

Hence

$\mathsf{d}_b(\mathrm{h}^*, [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}}) = 0$, and $\mathrm{h}^* \in [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}}$.

By the same above process, $\mathrm{h}^* \in [\Re_1(\mathrm{h}^*)]_{\alpha_{\Re_1(\mathrm{h}^*)}}$.

Therefore $\mathrm{h}^* \in [\Re_2(\mathrm{h}^*)]_{\alpha_{\Re_2(\mathrm{h}^*)}} \cap [\Re_1(\mathrm{h}^*)]_{\alpha_{\Re_1(\mathrm{h}^*)}}$. □

Example 3.2.3. Let $\mathsf{G} = [0, 1]$ accomplished with b-metric

$$\mathsf{d}_b(\zeta_1, \zeta_2) = |\zeta_1 - \zeta_2|^2,$$

then $(\mathsf{G}, \mathsf{d}_b)$ is a complete b-MS with $s = \frac{3}{2}$.

Let $P, \psi : [0, \infty) \longrightarrow [0, \infty)$ be defined as $\psi(\mathsf{t}) = \mathsf{t}$ and $P(\mathsf{t}) = \frac{99\mathsf{t}}{100}$ and $\alpha \in (0, 1]$.

Define $\Re_1, \Re_2 : \mathsf{G} \to \mathcal{F}(\mathsf{G})$ by,

$$\Re_1(\zeta_1)\mathsf{t} = \begin{cases} \alpha, & \text{if } 0 \leq \mathsf{t} \leq \frac{\zeta_1}{45},\\ \frac{\alpha}{2}, & \text{if } \frac{\zeta_1}{45} \leq \mathsf{t} \leq \frac{\zeta_1}{30},\\ \frac{\alpha}{3}, & \text{if } \frac{\zeta_1}{30} \leq \mathsf{t} \leq \frac{\zeta_1}{15},\\ \frac{\alpha}{5}, & \text{if } \frac{\zeta_1}{15} \leq \mathsf{t} \leq 1.\end{cases}$$

and

$$\Re_2(\zeta_2)\mathsf{t} = \begin{cases} \alpha, & \text{if } 0 \leq \mathsf{t} \leq \frac{\zeta_2}{45},\\ \frac{\alpha}{3}, & \text{if } \frac{\zeta_2}{45} \leq \mathsf{t} \leq \frac{\zeta_2}{30},\\ \frac{\alpha}{4}, & \text{if } \frac{\zeta_2}{30} \leq \mathsf{t} \leq \frac{\zeta_2}{15},\\ \frac{\alpha}{7}, & \text{if } \frac{\zeta_2}{15} \leq \mathsf{t} \leq 1.\end{cases}$$

Now for $\alpha_{\Re_2}$

and $\alpha_{\Re_1} = 1$ we have

$$[\Re_1(\zeta_1)]_\alpha = [0, \frac{\zeta_1}{45}], [\Re_2(\zeta_2)]_\alpha = [0, \frac{\zeta_2}{45}].$$

Hence

$$\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) = |\frac{\zeta_1}{45} - \frac{\zeta_2}{45}|^2 > 0 \text{ for } \zeta_1 \neq \zeta_2.$$

Also

$$\psi(\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}})) \leq P(\psi(\mathtt{M}(\zeta_1, \zeta_2))) + \mathfrak{L}\Xi(\zeta_1, \zeta_2),$$

with

$$\mathtt{M}(\zeta_1, \zeta_2) = \frac{1}{s}\max\Bigg\{\mathsf{d}_b(\zeta_1, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}), \mathsf{d}_b(\zeta_1, \zeta_2), \mathsf{d}_b(\zeta_2, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}), \frac{\{\mathsf{d}_b(\zeta_1, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) + \mathsf{d}_b(\zeta_2, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}})\}}{2s}\Bigg\}.$$

$$\begin{aligned}
&\mathtt{M}(\zeta_1, \zeta_2) \\
&= \frac{2}{3}\max\Bigg\{|\zeta_1 - \zeta_2|^2, |\zeta_1 - \frac{\zeta_1}{45}|^2, |\zeta_2 - \frac{\zeta_2}{45}|^2, \frac{\{|\zeta_1 - \frac{\zeta_2}{45}|^2 + |\zeta_2 - \frac{\zeta_1}{45}|^2\}}{3}\Bigg\}, \\
&\leq \frac{2}{3}|\zeta_1 - \zeta_2|^2,
\end{aligned}$$

and

$$\Xi(\zeta_1, \zeta_2) = \min\Bigg\{\mathsf{d}_b(\zeta_1, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}), \mathsf{d}_b(\zeta_2, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}), \mathsf{d}_b(\zeta_1, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}), \mathsf{d}_b(\zeta_2, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}})\Bigg\}.$$

This implies

$$\Xi(\zeta_1, \zeta_2) = \min\Bigg\{|\zeta_1 - \frac{\zeta_1}{45}|^2, |\zeta_2 - \frac{\zeta_2}{45}|^2, |\zeta_1 - \frac{\zeta_2}{45}|^2, |\zeta_2 - \frac{\zeta_1}{45}|^2)\Bigg\}.$$

Theorem 3.2.2 with $\mathfrak{L} = 1$ is satisfied, so $0 \in [\Re_1(0)]_\alpha \cap [\Re_2(0)]_\alpha$.

If we take $\mathfrak{L} = 0$ in Definition 3.2.1, then we obtain.

Corollary 3.2.4. Let $(\mathsf{G}, \mathsf{d}_b)$ be a complete b-MS, and let $\Re_1, \Re_2$ be the pair of (P, ψ) type almost contractive mapping satisfying:
$\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) > 0,$

$$\Rightarrow \psi(\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) \leq P(\psi(\mathtt{M}(\zeta_1, \zeta_2))),$$

where $\mathtt{M}(\zeta_1, \zeta_2)$ is defined same as in Definition 3.2.1. Then, $\exists$ h $\in \mathsf{G}$ such that $\mathrm{h} \in \{[\Re_1(\mathrm{h})]_{\alpha_{\Re_1(\mathrm{h})}} \cap [\Re_2(\mathrm{h})]_{\alpha_{\Re_2(\mathrm{h})}}\}$.

By taking $\Re_2 = \Re_1$ in Definition 3.2.1 we obtain.

Corollary 3.2.5. Assume $(\mathsf{G}, \mathsf{d}_b)$ be a complete b-MS and $\Re_1 : \mathsf{G} \to \mathcal{F}(\mathsf{G})$ be a FM if for each $\zeta_1 \in \mathsf{G}$ there exist $\alpha_{\Re_1(\zeta_1)} \in (0, 1]$ such that $[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}$ is a non empty and $\in \mathfrak{CB}(\mathsf{G})$, such that $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$,
$\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}) > 0$ implies

$$\psi(\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}})) \leq P(\psi(\mathtt{M}(\zeta_1, \zeta_2))) + \mathfrak{L}\Xi(\zeta_1, \zeta_2),$$

with

$$\begin{aligned}
&\mathtt{M}(\zeta_1, \zeta_2)\\
&= \frac{1}{s} \max\Bigg\{\mathsf{d}_b(\zeta_1, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}), \mathsf{d}_b(\zeta_1, \zeta_2), \mathsf{d}_b(\zeta_2, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}),\\
&\frac{\{\mathsf{d}_b(\zeta_1, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}) + \mathsf{d}_b(\zeta_2, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}})\}}{2s}\Bigg\},
\end{aligned}$$

and

$$\begin{aligned}
\Xi(\zeta_1, \zeta_2) = \min\Big\{&\mathsf{d}_b(\zeta_1, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}), \mathsf{d}_b(\zeta_2, [\Re_1(\zeta_2)]_{\alpha_{\Re_1}(\zeta_2)}),\\
&\mathsf{d}_b(\zeta_1, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}), \mathsf{d}_b(\zeta_2, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}})\Big\},
\end{aligned}$$

here $P \in \upsilon,\ \ \psi \in \kappa$ and $\mathfrak{L} \geq 0$.
Then $\exists$ h $\in \mathsf{G}$ such that $\mathrm{h} \in \{[\Re_1(\mathrm{h})]_{\alpha_{\Re_1(\mathrm{h})}}\}$.

Corollary 3.2.6. Let $(\mathsf{G}, \mathsf{d}_b)$ be a complete b-MS. Let $\hat{\Re}_1, \hat{\Re}_2 : \mathsf{G} \longrightarrow \mathcal{F}(\mathsf{G})$ be the pair of FM. If $\hat{\Re}_2(\zeta_1)$ and $\hat{\Re}_1(\zeta_2)$ are nonempty and $\in \mathfrak{CB}(\mathsf{G})$, such that $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$,
$\mathrm{H}(\hat{\Re}_1(\zeta_1), \hat{\Re}_2(\zeta_2)) > 0$ implies

$$\psi(\mathrm{H}(\hat{\Re}_1(\zeta_1), \hat{\Re}_2(\zeta_2)) \leq P(\psi(\mathtt{M}(\zeta_1, \zeta_2))) + \mathfrak{L}\Xi(\zeta_1, \zeta_2),$$

with

$$\begin{aligned}&\mathtt{M}(\zeta_1, \zeta_2)\\ &= \frac{1}{s}\max\left\{\mathsf{d}_b(\zeta_1, \hat{\Re}_1(\zeta_1), \mathsf{d}_b(\zeta_2, \hat{\Re}_2(\zeta_2), \mathsf{d}_b(\zeta_1, \zeta_2),\right.\\ &\qquad \left.\frac{\{\mathsf{d}_b(\zeta_1, \hat{\Re}_2(\zeta_2) + \mathsf{d}_b(\zeta_2, \hat{\Re}_1(\zeta_1)\}}{2s}\right\},\end{aligned}$$

and

$$\begin{aligned}\Xi(\zeta_1, \zeta_2) = \min\Big\{&\mathsf{d}_b(\zeta_1, \hat{\Re}_1(\zeta_1), \mathsf{d}_b(\zeta_2, \hat{\Re}_2(\zeta_2),\\ &\mathsf{d}_b(\zeta_1, \hat{\Re}_2(\zeta_2), \mathsf{d}_b(\zeta_2, \hat{\Re}_1(\zeta_1)\Big\},\end{aligned}$$

here $P \in \upsilon$, $\psi \in \kappa$ and $\mathfrak{L} \geq 0$.
Then $\exists\ \mathrm{h}^* \in \mathsf{G}$ such that $\Re_1(\mathrm{h}^*)(\mathrm{h}^*) \geq \Re_1(\mathrm{h}^*)(\mathrm{h})$ and $\Re_2(\mathrm{h}^*)(\mathrm{h}^*) \geq \Re_2(\mathrm{h}^*)(\mathrm{h})$ $\forall\ \mathrm{h} \in \mathsf{G}$.

Proof. By Theorem 3.2.2 $\exists\ \mathrm{h}^* \in \mathsf{G}$ so that $\mathrm{h}^* \in \Re_1(\mathrm{h}^*) \cap \Re_2(\mathrm{h}^*)$ with the help of Lemma 3.1.8 $\Re_1(\mathrm{h}^*)(\mathrm{h}^*) \geq \Re_1(\mathrm{h}^*)(\mathrm{h})$ and $\Re_2(\mathrm{h}^*)(\mathrm{h}^*) \geq \Re_1(\mathrm{h}^*)(\mathrm{h})$ $\forall\ \mathrm{h} \in \mathsf{G}$. □

In Corrolary 3.2.6 by taking $\mathfrak{L} = 0$ we obtain:

Corollary 3.2.7. Let $(\mathsf{G}, \mathsf{d}_b)$ be a complete b-MS. Let $\hat{\Re}_1, \hat{\Re}_2 : \mathsf{G} \longrightarrow \mathcal{F}(\mathsf{G})$ be the pair of FM. If $\hat{\Re}_2(\zeta_1)$ and $\hat{\Re}_1(\zeta_2)$ are nonempty and $\in \mathfrak{CB}(\mathsf{G})$, such that $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$,
$\mathrm{H}(\hat{\Re}_1(\zeta_1), \hat{\Re}_2(\zeta_2)) > 0$ implies

$$\psi(\mathrm{H}(\hat{\Re}_1(\zeta_1), \hat{\Re}_2(\zeta_2)) \leq P(\psi(\mathtt{M}(\zeta_1, \zeta_2))),$$

where $\mathtt{M}(\zeta_1, \zeta_2)$ is defined same as in Corollary 3.2.6, $P \in \upsilon$, $\psi \in \kappa$ and $\mathfrak{L} \geq 0$.

Then there exists $h^* \in \mathsf{G}$ such that $\Re_1(h^*)(h^*) \geq \Re_1(h^*)(h)$ and $\Re_2(h^*)(h^*) \geq \Re_1(h^*)(h) \ \ \forall\, h \in \mathsf{G}$

Theorem 3.2.8. Assume $(\mathsf{G}, \eth)$ being a complete b-metric linear space and let $\Re_1, \Re_2 : \mathsf{G} \to \mathfrak{D}(\mathsf{G})$ be the pair of FM.

Assume there are $P \in \upsilon$ and $\psi \in \kappa$ and $\mathfrak{L} \geq 0$ such that $\forall\, \zeta_1, \zeta_2 \in \mathsf{G}$, $\eth(\Re_1(\zeta_1), \Re_2(\zeta_2)) > \ 0$ implies,

$$\psi(\eth(\Re_1(\zeta_1), \Re_2(\zeta_2) \leq P(\psi(\mathtt{M}(\zeta_1, \zeta_2))) + \mathfrak{L}\Xi(\zeta_1, \zeta_2),$$

with

$$\mathtt{M}(\zeta_1, \zeta_2) = \frac{1}{s} \max \Bigg\{ Q(\zeta_1, \Re_1(\zeta_1)), Q(\zeta_1, \zeta_2), Q(\zeta_2, \Re_2(\zeta_2)), \\ \frac{\{Q(\zeta_1, \Re_2(\zeta_2)) + Q(\zeta_2, \Re_1(\zeta_1))\}}{2s} \Bigg\},$$

and

$$\Xi(\zeta_1, \zeta_2) = \min \Big\{ Q(\zeta_1, \Re_1(\zeta_1)), Q(\zeta_2, \Re_2(\zeta_2)), \\ Q(\zeta_1, \Re_2(\zeta_2)), Q(\zeta_2, \Re_1(\zeta_1)) \Big\},$$

then $\exists\, h \in \mathsf{G}$ such that $\{h\} \subset \Re_1(h)$ and $\{h\} \subset \Re_2(h)$.

Proof. Consider $\zeta_1 \in \mathsf{G}$, by using Lemma 3.1.6 there is $\zeta_2 \in \mathsf{G}$ so that $\zeta_2 \in [\Re_1(\zeta_1)]_1$. Similarly one can obtain $\zeta_3 \in [\Re_2(\zeta_1)]_1$. This means that for each $\zeta_1 \in \mathsf{G}$, $[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}$ and $[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}$ are nonempty and $\in \mathfrak{CB}(\mathsf{G})$. As $\alpha(\zeta_1) = \alpha(\zeta_1) = 1$ by the definition of $\eth$ for FSs, we have

$$H([\Re_1(\zeta_1)]_{\alpha(\zeta_1)}, [\Re_2(\zeta_2)]_{\alpha(\zeta_2)}) \leq \eth(\Re_1(\zeta_1), \Re_2(\zeta_2)) \quad \forall\, \zeta_1, \zeta_2 \in \mathsf{G}.$$

Since ψ is non-decreasing,

$$\psi(H([\Re_1(\zeta_1)]_{\alpha(\zeta_1)}, [\Re_2(\zeta_2)]_{\alpha(\zeta_2)}) \leq \psi(\eth(\Re_1(\zeta_1), \Re_2(\zeta_2))) \\ \leq P(\psi(\mathtt{M}(\zeta_1, \zeta_2))) + \mathfrak{L}\Xi(\zeta_1, \zeta_2).$$

Since $[\Re_2(\zeta_1)]_1 \subset [\Re_2(\zeta_1)]_{\alpha(\zeta_1)}$, therefore

$$\eth(\zeta_1, [\Re_2(\zeta_1)]_{\alpha(\zeta_1)}) \leq \mathsf{d}_b(\zeta_1, [\Re_2(\zeta_1)]_1) \text{ for each } \alpha \in (0, 1].$$

It yields that

$$\mathrm{Q}(\zeta_1, [\Re_1(\zeta_1)]_{\alpha(\zeta_1)}) \leq \mathsf{d}_b(\zeta_1, [\Re_1(\zeta_1)]_1).$$

Similarly,

$$\mathrm{Q}(\zeta_1, [\Re_2(\zeta_1)]_{\alpha(\zeta_1)}) \leq \mathsf{d}_b(\zeta_1, [\Re_2(\zeta_1)]_1).$$

It yields $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$,

$$\psi(\mathrm{H}([\Re_1(\zeta_1)]_1, [\Re_2(\zeta_2)]_1) \leq P(\psi(\mathtt{M}(\zeta_1, \zeta_2))) + \mathfrak{L}\Xi(\zeta_1, \zeta_2),$$

with

$$\begin{aligned} &\mathtt{M}(\zeta_1, \zeta_2) \\ &= \frac{1}{s} \max \Bigg\{ \mathsf{d}_b(\zeta_1, [\Re_1(\zeta_1)]_1), \mathsf{d}_b(\zeta_2, [\Re_2(\zeta_2)]_1), \mathsf{d}_b(\zeta_1, \zeta_2), \\ &\quad \frac{\{\mathsf{d}_b(\zeta_2, [\Re_1(\zeta_1)]_1) + \mathsf{d}_b(\zeta_1, [\Re_2(\zeta_2)]_1)\}}{2s} \Bigg\}, \end{aligned}$$

and

$$\begin{aligned} \Xi(\zeta_1, \zeta_2) = \min \Big\{ &\mathsf{d}_b(\zeta_1, [\Re_1(\zeta_1)]_1), \mathsf{d}_b(\zeta_2, [\Re_2(\zeta_2)]_1), \\ &\mathsf{d}_b(\zeta_1, [\Re_2(\zeta_2)]_1), \mathsf{d}_b(\zeta_2, [\Re_1(\zeta_1)]_1) \Big\}, \end{aligned}$$

then by Theorem 3.2.2 one can obtain there is $\mathrm{h} \in \mathsf{G}$ such that $\mathrm{h} \in \{[\Re_1(\mathrm{h})]_1 \cap [\Re_2(\mathrm{h})]_1\}$. □

By taking $\mathfrak{L} = 0$ in Theorem 3.2.8 we get:

Corollary 3.2.9. Assume $(\mathsf{G}, \eth)$ being a complete b-metric linear space and let $\Re_1, \Re_2 : \mathsf{G} \to \mathfrak{D}(\mathsf{G})$ be the pair of FM. Assume there are $P \in \upsilon$ and $\psi \in \kappa$ and $\mathfrak{L} \geq 0$ such that $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$, $\eth(\Re_1(\zeta_1), \Re_2(\zeta_2)) > 0$ implies,

$$\psi(\eth(\Re_1(\zeta_1), \Re_2(\zeta_2) \leq P(\psi(\mathtt{M}(\zeta_1, \zeta_2))),$$

where $\mathtt{M}(\zeta_1,\zeta_2)$ is defined same as in Theorem 3.2.8. Then $\exists$ h $\in$ G such that $\{\text{h}\} \subset \Re_1(\text{h})$ and $\{\text{h}\} \subset \Re_2(\text{h})$.

Remark 3.2.10. If we take $s = 1$, the results of Ameer et al. [80] become a special case of our results.

3.3 Some Consequences

This section will present a few consequences of our results on MMs.

Theorem 3.3.1. Consider the MMs $\mathcal{K}, \mathcal{J} : \mathsf{G} \longrightarrow \mathfrak{CB}(\mathsf{G})$. Suppose that $\forall\, \zeta_1, \zeta_2 \in \mathsf{G}$, $\mathrm{H}(\mathcal{K}(\zeta_1), \mathcal{J}(\zeta_2)) > 0$

$$\Rightarrow \psi(\mathrm{H}(\mathcal{K}(\zeta_1), \mathcal{J}(\zeta_2))) \leq P(\psi(\mathtt{M}(\zeta_1,\zeta_2))) + \mathfrak{L}\Xi(\zeta_1,\zeta_2),$$

with,

$$\begin{aligned}
&\mathtt{M}(\zeta_1,\zeta_2)\\
&= \frac{1}{s}\max\Bigg\{\mathsf{d}_b(\zeta_1,\mathcal{K}(\zeta_1)), \mathsf{d}_b(\zeta_1,\zeta_2), \mathsf{d}_b(\zeta_2,\mathcal{J}(\zeta_2)),\\
&\qquad \frac{\{\mathsf{d}_b(\zeta_2,\mathcal{J}(\zeta_2)) + \mathsf{d}_b(\zeta_2,\mathcal{K}(\zeta_1))\}}{2s}\Bigg\},
\end{aligned}$$

and

$$\Xi(\zeta_1,\zeta_2) = \min\Big\{\mathsf{d}_b(\zeta_1,\mathcal{K}(\zeta_1)), \mathsf{d}_b(\zeta_2,\mathcal{J}(\zeta_2)), \mathsf{d}_b(\zeta_1,\mathcal{J}(\zeta_2)), \mathsf{d}_b(\zeta_2,\mathcal{K}(\zeta_1))\Big\},$$

here, $P \in \upsilon$, $\psi \in \kappa$ and $\mathfrak{L} \geq 0$.
Then, $\exists$ h $\in \mathsf{G}$ such that h $\in \{\mathcal{K}(\text{h}) \cap \mathcal{J}(\text{h})\}$.

Proof. Take $\alpha : \mathsf{G} \longrightarrow (0,1]$ with $\Re_1, \Re_2$ be the pair of FM defined by,

$$\Re_1(\zeta_1)(\text{t}) = \begin{cases} \alpha(\zeta_1), & \text{if } \text{t} \in \mathcal{K}(\zeta_1),\\ 0, & \text{if } \text{t} \notin \mathcal{K}(\zeta_1),\end{cases}$$

and

$$\Re_2(\zeta_1)(\mathrm{t}) = \begin{cases} \alpha(\zeta_1), & \text{if } \mathrm{t} \in \mathcal{J}(\zeta_1), \\ 0, & \text{if } \mathrm{t} \notin \mathcal{J}(\zeta_1). \end{cases}$$

Then,

$$[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}} = \Big\{\mathrm{t} : \Re_1(\zeta_1)(\mathrm{t}) \geq \alpha(\zeta_1)\Big\} = \mathcal{K}(\zeta_1),$$
$$[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}} = \Big\{\mathrm{t} : \Re_2(\zeta_1)(\mathrm{t}) \geq \alpha(\zeta_1)\Big\} = \mathcal{J}(\zeta_1).$$

Thus, by Theorem 3.2.2 there exists $\mathrm{h} \in \mathsf{G}$ such that

$$\mathrm{h} \in \{[\Re_1(\mathrm{h})]_{\alpha_{\Re_1(\mathrm{h})}} \cap [\Re_2(\mathrm{h})]_{\alpha_{\Re_2(\mathrm{h})}}\} = \mathcal{K}(\mathrm{h}) \cap \mathcal{J}(\mathrm{h}).$$

□

Corollary 3.3.2. Assume $(\mathsf{G}, \mathsf{d}_b)$ be a complete b-MS, and let $\Re_1, \Re_2$ be the pair of FM from G into $\mathcal{F}(\mathsf{G})$. If for each $\zeta_1 \in \mathsf{G}$, $\exists\ \alpha_{\Re_1(\zeta_1)}, \alpha_{\Re_2(\zeta_1)} \in (0, 1]$ such that $[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}$, $[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}} \in \mathfrak{CB}(\mathsf{G})$ are non empty,

$$\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) \leq \lambda(\mathtt{M}(\zeta_1, \zeta_2)) + \mathfrak{L}\Xi(\zeta_1, \zeta_2),$$

where $\mathtt{M}(\zeta_1, \zeta_2)$ and $\Xi(\zeta_1, \zeta_2)$ are defined same as in Definition 3.2.1, $\lambda \in (0, 1)$ and $\mathfrak{L} \geq 0\ \forall\ \zeta_1, \zeta_2 \in \mathsf{G}$.

Then, there exist $\mathrm{h} \in \mathsf{G}$ such that $\mathrm{h} \in \{[\Re_1(\mathrm{h})]_{\alpha_{\Re_1(\mathrm{h})}} \cap [\Re_2(\mathrm{h})]_{\alpha_{\Re_2(\mathrm{h})}}\}$.

Proof. Proof follows by taking $P(t) = \lambda(t)$ and $\psi(t) = t$ in Theorem 3.2.2. □

In the following corollary, we have used generalized $\mathtt{F}$-contraction for FM and proved FP results as generalizations of Wardowski $\mathtt{F}$-contraction.

Corollary 3.3.3. Let $(\mathsf{G}, \mathsf{d}_b)$ be a complete b-MS, and $\Re_1, \Re_2 : \mathsf{G} \to \mathcal{F}(\mathsf{G})$ be the pair of FM. If for each $\zeta_1 \in \mathsf{G}$, $\exists\ \alpha_{\Re_1(\zeta_1)}, \alpha_{\Re_2(\zeta_1)} \in (0, 1]$ such that $[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}$, $[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}} \in \mathfrak{CB}(\mathsf{G})$ are non empty. Suppose that there are $\mathtt{F} \in \mathsf{F}, \tau > 0$ and $\mathfrak{L} \geq 0$ such that $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$,

$\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) > 0$

$$\Rightarrow \tau + \mathtt{F}((\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}})) \leq \mathtt{F}(\mathtt{M}(\zeta_1, \zeta_2)) + \mathfrak{L}\Xi(\zeta_1, \zeta_2),$$

$\mathtt{M}(\zeta_1, \zeta_2)$ and $\Xi(\zeta_1, \zeta_2)$ are defined same as in Definition 3.2.1.

Then, $\exists$ h $\in \mathsf{G}$ such that h $\in \{[\Re_1(\mathrm{h})]_{\alpha_{\Re_1(\mathrm{h})}} \cap [\Re_2(\mathrm{h})]_{\alpha_{\Re_2(\mathrm{h})}}\}$.

Proof. Proof follows by taking $P(t) = e^{-\tau}t$ and $\psi(t) = e^t$ in Theorem 3.2.2. □

If we take $\mathfrak{L} = 0$ in Corollary 3.3.3 then we obtain

Corollary 3.3.4. Let $(\mathsf{G}, \mathsf{d}_b)$ be a complete b-MS, and $\Re_1, \Re_2 : \mathsf{G} \to \mathcal{F}(\mathsf{G})$ be the pair of FM. If for each $\zeta_1 \in \mathsf{G}$, $\exists$ $\alpha_{\Re_1(\zeta_1)}, \alpha_{\Re_2(\zeta_1)} \in (0, 1]$ such that $[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}$, $[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}} \in \mathfrak{CB}(\mathsf{G})$ are non empty.

Suppose there are

$\mathtt{F} \in \mathsf{F}$ and $\tau > 0$ such that $\forall$ $\zeta_1, \zeta_2 \in \mathsf{G}$,

$\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) > 0$,

$$\Rightarrow \tau + \mathtt{F}((\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}})) \leq \mathtt{F}(\mathtt{M}(\zeta_1, \zeta_2)),$$

$\mathtt{M}(\zeta_1, \zeta_2)$ is defined same as in Definition 3.2.1.
Then, $\exists$ h $\in \mathsf{G}$ such that h $\in \{[\Re_1(\mathrm{h})]_{\alpha_{\Re_1(\mathrm{h})}} \cap [\Re_2(\mathrm{h})]_{\alpha_{\Re_2(\mathrm{h})}}\}$.

Corollary 3.3.5. Let $(\mathsf{G}, \mathsf{d}_b)$ be a complete b-MS, and let $\Re_1, \Re_2 : \mathsf{G} \to \mathcal{F}(\mathsf{G})$ be the pair of FM. If for each $\zeta_1 \in \mathsf{G}$, $\exists$ $\alpha_{\Re_1(\zeta_1)}, \alpha_{\Re_2(\zeta_1)} \in (0, 1]$ such that $[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}$, $[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}} \in \mathfrak{CB}(\mathsf{G})$ are non empty. Suppose that there is $\mathfrak{L} \geq 0$ such that $\forall$ $\zeta_1, \zeta_2 \in \mathsf{G}$,
$\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) > 0$,

$$\Rightarrow \psi(\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}})) \leq \Pi(\mathtt{M}(\zeta_1, \zeta_2))(\mathtt{M}(\zeta_1, \zeta_2)) + \mathfrak{L}\Xi(\zeta_1, \zeta_2),$$

where $\mathtt{M}(\zeta_1, \zeta_2)$ and $\Xi(\zeta_1, \zeta_2)$ are defined same as in Definition 3.2.1.
Then $\exists$ h $\in \mathsf{G}$ such that h $\in \{[\Re_1(\mathrm{h})]_{\alpha_{\Re_1(\mathrm{h})}} \cap [\Re_2(\mathrm{h})]_{\alpha_{\Re_2(\mathrm{h})}}\}$. Also $\Pi : [0, \infty) \longrightarrow [0, \infty)$ with $\lim_{\zeta \longrightarrow \mathrm{t}^+} \Pi(\zeta) < 1$ for each $\mathrm{t} \in (0, \infty)$.

Proof. Proof follows by taking $P(t) = \Pi(t)t$ and $\psi(t) = t$ in Theorem 3.2.2. □

Corollary 3.3.6. Let $(\mathsf{G}, \mathsf{d}_b)$ be a complete b-MS and $\Re_1, \Re_2 : \mathsf{G} \to \mathcal{F}(\mathsf{G})$ be the pair of FM. If for each $\zeta_1 \in \mathsf{G}$, $\exists\ \alpha_{\Re_1(\zeta_1)}, \alpha_{\Re_2(\zeta_1)} \in (0, 1]$ such that $[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}$, $[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}} \in \mathfrak{CB}(\mathsf{G})$ are non empty. Assume $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$ there are $\theta \in \Theta$ and $\mathsf{k} \in (0, 1)$ such that,
$\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) > 0,$

$$\Rightarrow \theta(\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}})) \leq \theta[\mathtt{M}(\zeta_1, \zeta_2)]^{\mathsf{k}},$$

$\mathtt{M}(\zeta_1, \zeta_2)$ is defined same as in Definition 3.2.1. Then $\exists\ \mathrm{h} \in \mathsf{G}$ such that $\mathrm{h} \in \{[\Re_1(\mathrm{h})]_{\alpha_{\Re_1(\mathrm{h})}} \cap [\Re_2(\mathrm{h})]_{\alpha_{\Re_2(\mathrm{h})}}\}$.

Proof. Proof follows by taking $P(t) = \ln(t)^{\mathsf{k}}$, $\psi(t) = \ln(t)$ and $\mathfrak{L} = 0$ in Theorem 3.2.2. □

3.4 Application

This section provides an application of our results for the existence of a solution to a second-order non-linear boundary value problem. Consider the following second-order non-linear boundary value problem:

$$\begin{cases} \zeta_1''(\mathfrak{a}) = \mathtt{K}(\mathfrak{a}, \zeta_1(\mathfrak{a}), \zeta_1'(\mathfrak{a})), & \mathfrak{a} \in [0, a],\ a > 0, \\ \zeta_1(\mathfrak{a}_1) = \mathsf{f}_1, & \mathfrak{a}_1 \in [0, a], \\ \zeta_1(\mathfrak{a}_2) = \mathsf{f}_2, & \mathfrak{a}_2 \in [0, a]. \end{cases} \tag{3.13}$$

where $\mathtt{K} : [0, a] \times \mathfrak{D}(\mathsf{G}) \times \mathfrak{D}(\mathsf{G}) \longrightarrow \mathfrak{D}(\mathsf{G})$ is a continuous function.
$\zeta_1(\mathfrak{a}) = \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a}, \varrho)\mathtt{K}(\mathfrak{a}, \zeta_1(\varrho), \zeta_1'(\varrho))\, d\varrho\ + \Pi(\mathfrak{a}), \mathfrak{a} \in [0, a],$
where Green's function $\mathfrak{J}$ for (3.13) is given by

$$\mathfrak{J}(\mathfrak{a}, \varrho) = \begin{cases} \dfrac{(\mathfrak{a}_2 - \mathfrak{a})(\varrho - \mathfrak{a})}{\mathfrak{a}_2 - \mathfrak{a}_1}, & \text{if } \mathfrak{a}_1 \leq \varrho \leq \mathfrak{a} \leq \mathfrak{a}_2, \\ \dfrac{(\mathfrak{a}_2 - \varrho)(\mathfrak{a} - \mathfrak{a}_1)}{\mathfrak{a}_2 - \mathfrak{a}_1}, & \text{if } \mathfrak{a}_1 \leq \mathfrak{a} \leq \varrho \leq \mathfrak{a}_2, \end{cases}$$

and $\Pi(\mathfrak{a})$ satisfies $\Pi''(\mathfrak{a}) = 0, \Pi(\mathfrak{a}_1) = \zeta_1$, and $\Pi(\mathfrak{a}_2) = \zeta_2$. Recall some properties of $\mathfrak{J}(\mathfrak{a}, \varrho)$. Particularly,

$$\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |\mathfrak{J}(\mathfrak{a}, \varrho)| d\varrho \ \leq \frac{(\mathfrak{a}_2 - \mathfrak{a}_1)^2}{8},$$

and

$$\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |\mathfrak{J}_{\mathfrak{a}}(\mathfrak{a}, \varrho)| d\varrho \ \leq \frac{(\mathfrak{a}_2 - \mathfrak{a}_1)}{2}.$$

We will find existence of a common FP of pair of operators given as,

$$\begin{aligned}
\Re_1(\zeta_1)(\mathfrak{a}) &= \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a}, \varrho) \mathtt{K}_1(\mathfrak{a}, \zeta_1(\varrho), \zeta_1'(\varrho))\, d\varrho \ + \Pi(\mathfrak{a}), \mathfrak{a} \in [0, a], \\
\Re_2(\zeta_2)(\mathfrak{a}) &= \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a}, \varrho) \mathtt{K}_2(\mathfrak{a}, \zeta_2(\varrho), \zeta_2'(\varrho))\, d\varrho \ + \Pi(\mathfrak{a}), \mathfrak{a} \in [0, a],
\end{aligned}$$

where $\mathtt{K}_1, \mathtt{K}_2 \in \mathsf{C}([0, a] \times \mathfrak{D}(\mathsf{G}) \times \mathfrak{D}(\mathsf{G}), \mathfrak{D}(\mathsf{G}))$, $\zeta_1 \in \mathsf{C}([0, a], \mathfrak{D}(\mathsf{G}))$ and $\Pi \in \mathsf{C}^1([0, a], \mathfrak{D}(\mathsf{G}))$.

Theorem 3.4.1. Assume that

(ς_1): $\mathtt{K}_1, \mathtt{K}_2 : [0, a] \times \mathfrak{D}(\mathsf{G}) \times \mathfrak{D}(\mathsf{G}) \longrightarrow \mathfrak{D}(\mathsf{G})$ are increasing in their second and third variables,

(ς_2): There is $\zeta_0 \in \mathsf{C}^1([0, a], \mathfrak{D}(\mathsf{G}))$ such that for $\mathfrak{a} \in [0, a]$, we have

$$\zeta_0(\mathfrak{a}) = \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a}, \varrho) \mathtt{K}_1(\mathfrak{a}, \zeta_0(\varrho), \zeta_0'(\varrho))\, d\varrho \ + \Pi(\mathfrak{a}), \mathfrak{a} \in [0, a].$$

(ς_3): There is $\mathfrak{a} \in [0, a]$, such that

$$\begin{aligned}
|\mathtt{K}_1(\mathfrak{a}, \zeta_1(\varrho), \zeta_1'(\varrho)) \ - \mathtt{K}_2(\mathfrak{a}, \zeta_2(\varrho), \zeta_2'(\varrho)) \ | &= |\zeta_1''(\varrho) - \zeta_2''(\varrho)|, \\
&\leq |\zeta_1(\varrho) - \zeta_2(\varrho)|,
\end{aligned}$$

for all $\zeta_1, \zeta_2 \in \mathsf{C}([0, a], \mathfrak{D}(\mathsf{G}))$ with $\mathtt{K}_1(.,.,.) \neq \mathtt{K}_2(.,.,.)$.

(ς_4): For $\mathfrak{a}_1, \mathfrak{a}_2 \in [0, a]$, we have,

$$\frac{(\mathfrak{a}_2 - \mathfrak{a}_1)^2}{8} + \frac{(\mathfrak{a}_2 - \mathfrak{a}_1)}{2} < \frac{9}{10}.$$

(ς_5): If $\zeta_1, \zeta_2 \in \mathsf{C}^1([0,a], \mathfrak{D}(\mathsf{G}))$ is comparable, then every $i \in [\Re_2(\zeta_1)]_1$ and $j \in [\Re_2(\zeta_2)]_1$ are comparable.

Then,

$$\zeta_1(\mathfrak{a}) = \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a}, \varrho)\mathtt{K}_1(\mathfrak{a}, \zeta_1(\varrho), \zeta_1'(\varrho))\, d\varrho \ + \Pi(\mathfrak{a}), \mathfrak{a} \in [0,a],$$

and

$$\zeta_2(\mathfrak{a}) = \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a}, \varrho)\mathtt{K}_2(\mathfrak{a}, \zeta_2(\varrho), \zeta_2'(\varrho))\, d\varrho \ + \Pi(\mathfrak{a}), \mathfrak{a} \in [0,a],$$

has a common solution in $\mathsf{C}^1([\mathfrak{a}_1, \mathfrak{a}_2], \mathfrak{D}(\mathsf{G}))$.

Proof. Consider $\mathsf{C} = \mathsf{C}^1([\mathfrak{a}_1, \mathfrak{a}_2], \mathfrak{D}(\mathsf{G}))$ with b-MS.

$$\mathsf{d}_b(\zeta_1, \zeta_2) = \max_{\varrho\in[0,a]} |\zeta_1(\varrho) - \zeta_2(\varrho)|^2.$$

Note that d_b is a complete b-MS. Let $\Re_2, \Re_1 : \mathsf{C} \longrightarrow \mathsf{C}$ be two integral operators as defined above. Clearly, $\Re_1$ and $\Re_2$ are well defined. Since $\mathtt{K}_1$ and $\mathtt{K}_2$ with Π are continuous functions. Now h^* is a solution if and only if h^* is a common FP for the pair of FM $(\Re_2, \Re_1)$. By (ς1), $\mathtt{K}_1, \mathtt{K}_2$ is increasing. Next for all $\zeta_1, \zeta_2 \in \mathsf{G}$ with $\mathtt{K}_1(.,.,.,) \neq \mathtt{K}_2(.,.,.,)$, by ($\varsigma_3$), we have,

$$\begin{aligned}
&|\Re_1(\zeta_1)(\mathfrak{a}) - \Re_2(\zeta_2)(\mathfrak{a})| \\
&= |(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a}, \varrho)\mathtt{K}_1(\mathfrak{a}, \zeta_1(\varrho), \zeta_1'(\varrho))\, d\varrho \ + \Pi(\mathfrak{a})) \\
&\quad - (\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a}, \varrho)\mathtt{K}_2(\mathfrak{a}, \zeta_2(\varrho), \zeta_2'(\varrho))\ d\varrho \ + \Pi(\mathfrak{a}))|, \qquad \mathfrak{a} \in [0,a], \\
&= |\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a}, \varrho)(\mathtt{K}_1(\mathfrak{a}, \zeta(\varrho), \zeta_1'(\varrho)) - \mathtt{K}_2(\mathfrak{a}, \zeta_2(\varrho), \zeta_2'(\varrho)))d\varrho|, \qquad \mathfrak{a} \in [0,a], \qquad (3.14) \\
&\leq |\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a}, \varrho)||\mathtt{K}_1(\mathfrak{a}, \zeta_1(\varrho), \zeta_1'(\varrho)) - \mathtt{K}_2(\mathfrak{a}, \zeta_2(\varrho), \zeta_2'(\varrho)))|d\varrho, \qquad \mathfrak{a} \in [0,a], \\
&\leq \max_{\varrho\in[0,a]} |\zeta_1(\varrho) - \zeta_2(\varrho)|^2 \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |\mathfrak{J}(\mathfrak{a}, \varrho)|d\varrho, \\
&\leq \mathsf{d}_b(\zeta_1, \zeta_2)\frac{(\mathfrak{a}_2 - \mathfrak{a}_1)^2}{8},
\end{aligned}$$

and

$$\begin{aligned}
&|\Re_1(\zeta_1')(\mathfrak{a}) - \Re_2(\zeta_2')(\mathfrak{a})| \\
&= |(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a},\varrho)\mathtt{K}_1(\mathfrak{a},\zeta_1(\varrho),\zeta_1'(\varrho))\,d\varrho\ + \Pi(\mathfrak{a})) \\
&\quad - (\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a},\varrho)\mathtt{K}_2(\mathfrak{a},\zeta_2(\varrho),\zeta_2'(\varrho))\ d\varrho\ + \Pi(\mathfrak{a}))|, \qquad \mathfrak{a} \in [0,a], \\
&= |\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a},\varrho)(\mathtt{K}_1(\varrho,\zeta(\varrho),\zeta_1'(\varrho)) - \mathtt{K}_2(\mathfrak{a},\zeta_2(\varrho),\zeta_2'(\varrho)))d\varrho|, \qquad \mathfrak{a} \in [0,a], \qquad (3.15)
\end{aligned}$$

$$\begin{aligned}
&\le |\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a},\varrho)||\mathtt{K}_1(\mathfrak{a},\zeta_1(\varrho),\zeta_1'(\varrho)) - \mathtt{K}_2(\mathfrak{a},\zeta_2(\varrho),\zeta_2'(\varrho)))|d\varrho, \qquad \mathfrak{a} \in [0,a], \\
&\le \max_{\varrho\in[0,a]} |\zeta_1(\varrho) - \zeta_2(\varrho)|^2 \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |\mathfrak{J}(\mathfrak{a},\varrho)|d\varrho, \\
&\le \mathsf{d}_b(\zeta_1,\zeta_2)\frac{(\mathfrak{a}_2 - \mathfrak{a}_1)}{2}.
\end{aligned}$$

From (3.14) and (3.15),

we easily obtain,

$$\begin{aligned}
\mathsf{d}_b(\Re_2(\zeta_1)(\mathfrak{a}), \Re_1(\zeta_2)(\mathfrak{a})) &\le \frac{(\mathfrak{a}_2 - \mathfrak{a}_1)^2}{8} + \frac{(\mathfrak{a}_2 - \mathfrak{a}_1)}{2}\mathsf{d}_b(\zeta_1,\zeta_2) \\
&\le \frac{9}{10}\mathsf{d}_b(\zeta_1,\zeta_2).
\end{aligned}$$

It implies that

$$e^{\mathsf{d}_b(\Re_2(\zeta_1)(\mathfrak{a}),\Re_1(\zeta_2)(\mathfrak{a}))} \le e^{\frac{9}{10}\mathsf{d}_b(\zeta_1,\zeta_2)} \le e^{\mathsf{d}_b(\zeta_1,\zeta_2)}.$$

Therefore,

$$\mathsf{d}_b(\Re_2(\zeta_1)(\mathfrak{a}), \Re_1(\zeta_2)(\mathfrak{a}))e^{\mathsf{d}_b(\Re_2(\zeta_1)(\mathfrak{a}),\Re_1(\zeta_2)(\mathfrak{a}))} \le \frac{9}{10}\mathsf{d}_b(\zeta_1,\zeta_2)e^{\mathsf{d}_b(\zeta_1,\zeta_2)}. \qquad (3.16)$$

Let $\psi, P : (0,\infty) \longrightarrow (0,\infty)$ defined by,

$$\begin{aligned}
\psi(\mathfrak{a}) &= \mathfrak{a}e^{\mathfrak{a}}, \quad \mathfrak{a} > 0, \\
P(\mathfrak{a}) &= \frac{9\mathfrak{a}}{10}, \quad \mathfrak{a} > 0,
\end{aligned}$$

respectively.

Thus we have from (3.16) we have

$$\psi(\mathtt{d}_b(\Re_2(\zeta_1)(\mathfrak{a}), \Re_1(\zeta_2)(\mathfrak{a}))) \leq P(\psi(\mathtt{d}_b(\zeta_1, \zeta_2))) \leq P(\psi(\mathtt{M}(\zeta_1, \zeta_2))).$$

So by Corollary 3.3.6 there exist a common FP h^* for $\Re_2$ and $\Re_1$. □

3.5 Conclusion

In this chapter, we extended the results of Ameer et al. [80] by introducing the (P, ψ) type contractive condition in complete b-MS. Some corollaries are presented to guarantee that our results are more general and are special cases of several existing results.

The continuity condition of the b-MS plays a crucial role in the current formulation. An open problem for future investigation is whether similar results can be established without imposing this condition.

Chapter 4

Fuzzy FP Results for Θ-Contraction

In this chapter we propose two innovative types of contraction mappings in the setting of double controlled MSs: the $\Theta - \texttt{FDCCM}$ and the $\Theta - \texttt{FDCCM}$. This development significantly enriches the current understanding of generalized contraction principles. We rigorously prove that each mapping admits a unique FP under appropriate conditions and include detailed examples to illustrate these findings. These findings are further applied to establish existence results for nonlinear differential equations. Several corollaries are also derived, revealing that our work unifies and generalizes numerous earlier results in the field.

4.1 Chapter Layout

The following structure is adopted in this chapter to facilitate a clearer understanding of the presented material:

- (i): The chapter commences with a collection of essential definitions that serve as a foundation for the forthcoming discussions.
- (ii): Section 4.2 introduces the concept of $\Theta - \texttt{FDCCM}$ in the framework of double controlled MS, inspired by the idea of almost contractions in MSs. A

FP theorem is established, followed by several corollaries derived from the main result. Additionally, an illustrative example is provided to support the theoretical findings.

(*iii*): In Section 4.3, the notion of $\Theta - \texttt{FAGDCCM}$ is proposed within the setting of double controlled MS, and FP results are established. Several corollaries are obtained, and an example is included to validate our results.

(*iv*): Section 4.4 explores MMs to further generalize and support the established FP results. Various corollaries are deduced based on different types of mappings and contractive conditions.

(*v*): In Section 4.5, the main theoretical results are applied to demonstrate the existence of a solution for a second-order nonlinear boundary value problem.

(*vi*): Section 4.6 concludes the chapter by offering a summary and key remarks to enhance comprehension of the overall work.

Motivated by the work of Azam et al. [75] we established the following definition in the context of double controlled MS as follows:

Definition 4.1.1. Assume that $\mathfrak{CB}(\mathsf{G})$ is the family of closed and bounded subsets of a double controlled MS $(\mathsf{G}, \mathsf{d}_{\sigma,\rho})$. For $\zeta_1 \in \mathsf{G}$ and $\mathsf{A}, \mathsf{B} \in \mathfrak{CB}(\mathsf{G})$, set

$$\mathsf{d}_{\sigma,\rho}(\zeta_1, \mathsf{B}) = \inf_{\mathsf{y}\in\mathsf{B}}(\mathsf{d}_{\sigma,\rho}(\zeta_1, \mathsf{y})).$$

Define $\mathrm{H}_{\sigma,\rho} : \mathfrak{CB}(\mathsf{G}) \times \mathfrak{CB}(\mathsf{G}) \longrightarrow \mathbb{R}^+$ as

$$\mathrm{H}_{\sigma,\rho}(\mathsf{A}, \mathsf{B}) = \max\left\{\sup_{\mathsf{a}\in\mathsf{A}} \mathsf{d}_{\sigma,\rho}(\mathsf{a}, \mathsf{B}), \sup_{\mathsf{b}\in\mathsf{B}} \mathsf{d}_{\sigma,\rho}(\mathsf{b}, \mathsf{A})\right\},$$

then $(\mathfrak{CB}(\mathsf{G}), \mathrm{H}_{\sigma,\rho})$ is named as Pompeiu-Hausdorff double controlled MS.

Example 4.1.2. Let $\mathsf{G} = \mathbb{R}$. Define $\mathsf{d}_{\sigma,\rho} : \mathsf{G} \times \mathsf{G} \longrightarrow [0, \infty)$ by

$$\mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2) = |\zeta_1 - \zeta_2|^2.$$

Then $(\mathsf{d}_{\sigma,\rho}, \mathsf{G})$ is a complete double controlled MS, where $\sigma, \rho : \mathsf{G} \times \mathsf{G} \longrightarrow (0, 1]$ be defined by

$$\sigma(\zeta_1, \zeta_2) = 2 + \zeta_2 + \zeta_1,$$

$$\rho(\zeta_2, \zeta_3) = 3 + \zeta_2 + \zeta_3.$$

For $\mathsf{A} = [0, 20]$ and let $\mathsf{B} = [22, 31]$. We define the distance from a point $\mathsf{a} \in \mathsf{A}$ to the set B as

$$\mathsf{d}_{\sigma,\rho}(\mathsf{a}, \mathsf{B}) = \inf_{\mathsf{b} \in \mathsf{B}} \mathsf{d}_{\sigma,\rho}(\mathsf{a}, \mathsf{b}),$$

which represents the smallest value among all distances from a to elements of B that is, the distance from a to the nearest point in B with respect to the metric $\mathsf{d}_{\sigma,\rho}$.
Consider $\mathsf{a} = 12$. Then

$$\mathsf{d}_{\sigma,\rho}(12, \mathsf{B}) = \inf_{22 \in \mathsf{B}} (\mathsf{d}_{\sigma,\rho}(12, 22)) = 100.$$

Observe that for each $\mathsf{a} \in \mathsf{A}$, the closest point in B minimizing the distance is always $\mathsf{b} = 22$. Thus,

$$\sup_{\mathsf{a} \in \mathsf{A}} \mathsf{d}_{\sigma,\rho}(\mathsf{a}, \mathsf{B}) = \sup\{\mathsf{d}_{\sigma,\rho}(\mathsf{a}, 22) \mid \mathsf{a} \in \mathsf{A}\}.$$

The point $\mathsf{a} = 0$ in A attains this supremum, giving

$$\sup\{\mathsf{d}_{\sigma,\rho}(\mathsf{a}, 22) \mid \mathsf{a} \in \mathsf{A}\} = \mathsf{d}_{\sigma,\rho}(0, 22) = 484.$$

Similarly, for $\mathsf{b} \in \mathsf{B}$, we have

$$\sup_{\mathsf{b} \in \mathsf{B}} \mathsf{d}_{\sigma,\rho}(\mathsf{b}, \mathsf{A}) = \sup\{\mathsf{d}_{\sigma,\rho}(\mathsf{b}, 20) \mid \mathsf{b} \in \mathsf{B}\}.$$

The maximum is attained at $\mathsf{b} = 31$ in B. Therefore

$$\begin{aligned} \sup\{\mathsf{d}_{\sigma,\rho}(\mathsf{b}, 20) \quad \mathsf{b} \in \mathsf{B}\} &= \sup\{\mathsf{d}_{\sigma,\rho}(31, 20) \quad 20 \in \mathsf{B}\} \\ &= 121. \end{aligned}$$

It follows that

$$\mathrm{H}_{\sigma,\rho}(\mathsf{A},\mathsf{B}) = \max\left\{\sup_{\mathsf{a}\in\mathsf{A}} \mathsf{d}_{\sigma,\rho}(\mathsf{a},\mathsf{B}), \sup_{\mathsf{b}\in\mathsf{B}} \mathsf{d}_{\sigma,\rho}(\mathsf{b},\mathsf{A})\right\}$$
$$= 484$$

In the context of double controlled MSs we can prove the following lemma by using the same procedure as in [93].

Lemma 4.1.3. Assume that $(\mathsf{G}, \mathsf{d}_{\sigma,\rho})$ is a complete double controlled MS, and $\mathsf{A}, \mathsf{B} \in \mathfrak{CB}(\mathsf{G})$. Then for each $\mathsf{a} \in \mathsf{A}$,

$$\mathsf{d}_{\sigma,\rho}(\mathsf{a},\mathsf{B}) \leq \mathrm{H}_{\sigma,\rho}(\mathsf{A},\mathsf{B}).$$

In 2008 Berinde et al. [52] introduced the notion of generalized almost contraction in the following manner.

Definition 4.1.4. For a MS (G, d), a mapping $\Re_2 : \mathsf{G} \longrightarrow \mathsf{G}$ is called generalized almost contraction if, $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$, the following condition holds:

$$\begin{aligned}&\mathsf{d}(\Re_2(\zeta_1), \Re_2(\zeta_2))\\ &\leq \lambda \mathsf{d}(\zeta_1,\zeta_2) + \mathfrak{L}\min[\mathsf{d}(\zeta_1,\Re_2(\zeta_2)), \mathsf{d}(\zeta_2,\Re_2(\zeta_1)), \mathsf{d}(\zeta_1,\Re_2(\zeta_1)), \mathsf{d}(\zeta_2,\Re_2(\zeta_2)),\end{aligned}$$

where $\mathfrak{L} \geq 0,\ 0 \leq \lambda < 1$.

4.2 Θ-Fuzzy Double Controlled Contraction Mapping

This section deals with common α-fuzzy FP result in the context of double controlled MS. Motivated by the idea of almost contraction on MS, we have introduced the notion of $\Theta - \mathtt{FDCCM}$ in double controlled MS as follows:

Definition 4.2.1. Assume that $(\mathsf{G}, \mathsf{d}_{\sigma,\rho})$ is a double controlled MS, then the pair $\Re_1, \Re_2 : \mathsf{G} \to \mathcal{F}(\mathsf{G})$ is $\Theta - \mathtt{FDCCM}$ if $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$ such that

$H_{\sigma,\rho}([\Re_1(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}, [\Re_1(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) > 0$ implies,

$$[\theta(H_{\sigma,\rho}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}))] \leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2))]^{\mathsf{r}},$$

$\theta \in \Theta$, $\mathsf{r} \in (0,1)$ and $[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}$, $[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}$ are non empty elements of $\mathfrak{CB}(\mathsf{G})$ and $\alpha_{\Re_1(\zeta_1)}, \alpha_{\Re_2(\zeta_1)} \in (0,1]$.

Theorem 4.2.2. Assume that $(\mathsf{G}, \mathsf{d}_{\sigma,\rho})$ is a complete double controlled MS, and $\Re_1, \Re_2 : \mathsf{G} \to \mathcal{F}(\mathsf{G})$ be the pair of $\Theta-$ `FDCCM` satisfying the

(i): $\sup_{m\geq 0} \lim_{n\longrightarrow\infty} \sigma(\zeta_{n+1}, \zeta_{n+2})\rho(\zeta_{n+1}, \zeta_m) < \frac{1}{\mathfrak{p}}$.

(ii): Both $\lim_{n\longrightarrow\infty} \sigma(\zeta, \zeta_n)$ and $\lim_{n\longrightarrow\infty} \rho(\zeta_n, \zeta)$ exist and are finite $\forall\ \zeta \in \mathsf{G}$ and the sequences $\{\zeta_{2n}\}$ and $\{\zeta_{2n+1}\}$ are defined as $\zeta_{2n} = {\Re_1}^n\zeta_0$ and $\zeta_{2n+1} = {\Re_2}^n\zeta_0$ for some $\zeta_0 \in \mathsf{G}$.

Then $\exists\ \zeta^* \in \mathsf{G}$ such that $\zeta^* \in \{[\Re_1(\zeta^*)]_{\alpha_{\Re_1(\zeta^*)}} \cap [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}\}$.

Proof. Let $\zeta_0 \in \mathsf{G}$. By hypothesis, $\exists\ \alpha_{\Re_1(\zeta_0)} \in (0,1]$ such that $[\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}$ is a non empty element of $\mathfrak{CB}(\mathsf{G})$. Let $\zeta_1 \in [\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}$ also $\exists\ \alpha_{\Re_2(\zeta_1)} \in (0,1]$ such that $[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}$ is non empty and $\in \mathfrak{CB}(\mathsf{G})$.
Since θ is strictly nondecreasing and by using Definition 4.1.1,

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_1, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}) \leq \theta(H_{\sigma,\rho}([\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}})). \tag{4.1}$$

Thus,

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_1, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}})) = \inf_{\mathsf{a}\in[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}} \theta(\mathsf{d}_{\sigma,\rho}(\zeta_1, \mathsf{a})).$$

Therefore,

$$\inf_{\mathsf{a}\in[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}} [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_1, \mathsf{a}))] \leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_0))]^{\mathsf{r}}.$$

So $\exists\ \zeta_2 \in [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}$ such that

$$\begin{aligned}\theta(\mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2) &\leq \theta(H_{\sigma,\rho}([\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}})) \\ &\leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0, \zeta_1))]^{\mathsf{r}}, \text{ from (4.1)}.\end{aligned} \tag{4.2}$$

Now $\exists\ \alpha_{\Re_1(\zeta_2)} \in (0,1]$ such that $[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}} \in \mathfrak{CB}(\mathsf{G})$ is a non empty set. By Lemma 4.1.1,

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_2, [\Re_1(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}})) \leq \theta(\mathrm{H}_{\sigma,\rho}([\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}). \tag{4.3}$$

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_2, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}})) = \inf_{\mathsf{q}\in[\Re_1(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}} \theta(\mathsf{d}_{\sigma,\rho}(\zeta_2, \mathsf{q})).$$

By hypothesis

$$\inf_{\mathsf{q}\in[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}} [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_2, \mathsf{q}))] \leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_2, \zeta_1))]^{\mathsf{r}}.$$

Thus, $\exists\ \zeta_3 \in [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}$ such that,

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_2, \zeta_3) \leq \theta(\mathrm{H}_{\sigma,\rho}([\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}})) \leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2))]^{\mathsf{r}}. \tag{4.4}$$

Using this procedure, we generate a sequence $\{\zeta_n\} \in \mathsf{G}$, with $\zeta_{2n+2} \in [\Re_2(\zeta_{2n+1})]_{\alpha_{\Re_2(\zeta_{2n+1})}}$ and $\zeta_{2n+1} \in [\Re_1(\zeta_{2n})]_{\alpha_{\Re_1(\zeta_{2n})}}$ such that

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{2n+1}, \zeta_{2n+2})) \leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{2n}, \zeta_{2n+1}))]^{\mathsf{r}}\ \forall\ n \in \mathbb{N}, \tag{4.5}$$

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{2n+2}, \zeta_{2n+3})) \leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{2n+1}, \zeta_{2n+2}))]^{\mathsf{r}}\ \forall\ n \in \mathbb{N}, \tag{4.6}$$

by combining (4.5) and (4.6) we may write

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})) \leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{n-1}, \zeta_n))]^{\mathsf{r}}\ \forall\ n \in \mathbb{N},$$

which further implies

$$\begin{aligned}
\theta(\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})) &\leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{n-1}, \zeta_n))]^{\mathsf{r}} \\
&\leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{n-2}, \zeta_{n-1}))]^{\mathsf{r}^2} \\
&\leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{n-3}, \zeta_{n-2}))]^{\mathsf{r}^3} \\
&\vdots \\
&\leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0, \zeta_1))]^{\mathsf{r}^n}.
\end{aligned}$$

As $\theta(t) > 1$, we have

$$1 < \theta(\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})) \leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0, \zeta_1))]^{\mathsf{r}^n}.$$

Sine $\theta \in \Theta$, by letting $n \longrightarrow \infty$, we arrive at

$$\lim_{n \longrightarrow \infty} \theta(\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})) = 1.$$

Thus, by Θ_2

$$\lim_{n \longrightarrow \infty} \mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1}) = 0^+.$$

In the view Θ_3, $\exists$ $\mathsf{p} \in (0, 1)$ and $\exists$ $\mathsf{M} \in (0, \infty]$ such that

$$\lim_{n \longrightarrow \infty} \frac{\theta(\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})) - 1}{\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})^{\mathsf{p}}} = \mathsf{M}.$$

Case 1: Let $0 < \mathsf{M} < \infty$ and $\frac{\mathsf{M}}{2} = \mathtt{C}$. Hence $\exists$ $n_0 \in \mathbb{N}$ such that $\forall\, n > n_0$

$$\Big| \frac{\theta(\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})) - 1}{\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})^{\mathsf{p}}} - \mathsf{M} \Big| \leq \mathtt{C}.$$

that is

$$\frac{\theta(\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})) - 1}{\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})^{\mathsf{p}}} \geq \mathsf{M} - \mathtt{C} = \mathtt{C}.$$

Then,

$$n[\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})]^{\mathsf{p}} \leq \frac{n}{\mathtt{C}}[\theta(\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1}))) - 1].$$

$$n[\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})]^{\mathsf{p}} \leq \frac{n}{\mathtt{C}}([\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0, \zeta_1))]^{\mathsf{p}^n} - 1).$$

By taking limit as $n \longrightarrow \infty$

$$\begin{aligned}
\lim_{n\longrightarrow\infty} n[\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})]^{\mathsf{p}} &\leq \lim_{n\longrightarrow\infty} \frac{n}{\mathsf{C}}([\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1))]^{\mathsf{p}^n}-1)\\
&\leq \frac{\lim_{n\longrightarrow\infty}([\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1))]^{\mathsf{p}^n}-1)}{\mathsf{C}\lim_{n\longrightarrow\infty}\frac{1}{n}}\\
&= \frac{\lim_{n\longrightarrow\infty}([\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1))]^{\mathsf{p}^n}-1)}{\mathsf{C}\lim_{n\longrightarrow\infty}\frac{1}{n}}\\
&= \frac{\lim_{n\longrightarrow\infty}\mathsf{p}^n\ln(\mathsf{p})\ln([\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1))])([\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1))]^{\mathsf{p}^n})}{\mathsf{C}\lim_{n\longrightarrow\infty}\frac{-1}{n^2}}\\
&= \frac{\ln(\mathsf{p})\ln([\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1))])\lim_{n\longrightarrow\infty}(-n^2)([\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1))]^{\mathsf{p}^n})}{\mathsf{C}\lim_{n\longrightarrow\infty}\mathsf{p}^{-n}}\\
&= \frac{\ln(\mathsf{p})\ln([\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1))])}{\mathsf{C}}\lim_{n\longrightarrow\infty}\frac{(-n^2)}{\mathsf{p}_1^n}\\
&= \frac{\ln(\mathsf{p})\ln([\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1))])}{\mathsf{C}}(0)\\
&= 0 \text{ (where } \mathsf{p}_1=\frac{1}{\mathsf{p}}),
\end{aligned}$$

which implies that

$$\lim_{n\longrightarrow\infty} n[\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})]^{\mathsf{p}} = 0. \tag{4.7}$$

Case 2: Let $\mathsf{M}=\infty$ and $\mathsf{C}>0$. Then $\exists \; n_1\in\mathbb{N}$ such that for all $n>n_1$

$$\mathsf{C} \leq \frac{\theta(\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1}))-1}{\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})^{\mathsf{p}}}.$$

$$\frac{\theta(\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})))-1}{\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})^{\mathsf{p}}} \geq \mathsf{M}-\mathsf{C}=\mathsf{C}.$$

Then,

$$n[\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})]^{\mathsf{p}} \leq \frac{n}{\mathsf{C}}[\theta(\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})))-1].$$

$$n[\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})]^{\mathsf{p}} \leq \frac{n}{\mathsf{C}}([\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1))]^{\mathsf{p}^n}-1).$$

By taking limit as n tends to infinity and using (4.7) we obtain

$$\lim_{n\longrightarrow\infty} n[\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})]^{\mathsf{p}} = 0.$$

In both cases $\frac{1}{\mathsf{c}} > 0$ we conclude that for any $\mathsf{M} \in (0,\infty]$ and $0 < \mathsf{p} < 1 \quad \exists$ some $n \in \mathbb{N}$,

where $\mathsf{N} = \max\{n_0, n_1\}$

such that $\exists \quad n_1$,

$$n[\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1)]^{\mathsf{p}} \leq 1 \text{ (for all } n > n_1).$$

$$\Rightarrow \mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1}) \leq \frac{1}{n^{\frac{1}{\mathsf{p}}}}. \tag{4.8}$$

Next, we show that $\{\zeta_n\}$ is a Cauchy sequence in G.

By triangular inequality, and $\forall\ m, n \geq n_0$

$$\begin{aligned}
&\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_m)\\
&\leq \sigma(\zeta_n,\zeta_{n+1})\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1}) + \rho(\zeta_{n+1},\zeta_m)\mathsf{d}_{\sigma,\rho}(\zeta_{n+1},\zeta_m)\\
&\leq \sigma(\zeta_n,\zeta_{n+1})\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1}) + \rho(\zeta_{n+1},\zeta_m)[\sigma(\zeta_{n+1},\zeta_{n+2})\mathsf{d}_{\sigma,\rho}(\zeta_{n+1},\zeta_{n+2})\\
&\qquad\qquad + \rho(\zeta_{n+2},\zeta_m)\mathsf{d}_{\sigma,\rho}(\zeta_{n+2},\zeta_m)],
\end{aligned}$$

which further implies

$$\begin{aligned}
&\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_m)\\
&\leq \sigma(\zeta_n,\zeta_{n+1})\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1}) + \rho(\zeta_{n+1},\zeta_m)\sigma(\zeta_{n+1},\zeta_{n+2})\mathsf{d}_{\sigma,\rho}(\zeta_{n+1},\zeta_{n+2})\\
&\qquad + \rho(\zeta_{n+1},\zeta_m)\rho(\zeta_{n+2},\zeta_m)[\sigma(\zeta_{n+2},\zeta_{n+3})\mathsf{d}_{\sigma,\rho}(\zeta_{n+2},\zeta_{n+3})\\
&\qquad + \rho(\zeta_{n+3},\zeta_m)\mathsf{d}_{\sigma,\rho}(\zeta_{n+3},\zeta_m)].\\
&\vdots
\end{aligned}$$

$$\begin{aligned}
&\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_m)\\
&\leq \sigma(\zeta_n,\zeta_{n+1})\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1}) + \sum_{i=1+n}^{m-2}\left(\prod_{\mathrm{j}=1+n}^{\mathrm{i}} \rho(\zeta_j,\zeta_m)\right)\sigma(\zeta_i,\zeta_{i+1})\mathsf{d}_{\sigma,\rho}(\zeta_i,\zeta_{i+1})
\end{aligned}$$

$$+\prod_{\mathrm{k}=1+n}^{m-1}\rho(\zeta_{\mathrm{k}},\zeta_m)\mathsf{d}_{\sigma,\rho}(\zeta_{m-1},\zeta_m). \tag{4.9}$$

$$\leq \sigma(\zeta_n,\zeta_{n+1})\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})+\sum_{i=1+n}^{m-2}\left(\prod_{\mathrm{j}=1+n}^{\mathrm{i}}\rho(\zeta_j,\zeta_m)\right)\sigma(\zeta_i,\zeta_{i+1})\mathsf{d}_{\sigma,\rho}(\zeta_i,\zeta_{i+1})$$

$$+\prod_{\mathrm{k}=1+n}^{m-1}\rho(\zeta_{\mathrm{k}},\zeta_m)\sigma(\zeta_{m-1},\zeta_m)\mathsf{d}_{\sigma,\rho}(\zeta_{m-1},\zeta_m). \tag{4.10}$$

$$\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_m)$$

$$\leq \sigma(\zeta_n,\zeta_{n+1})\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})+\sum_{i=1+n}^{m-1}\left(\prod_{\mathrm{j}=1+n}^{\mathrm{i}}\rho(\zeta_j,\zeta_m)\right)\sigma(\zeta_i,\zeta_{i+1})\mathsf{d}_{\sigma,\rho}(\zeta_i,\zeta_{i+1}).$$

Hence

$$\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_m)\leq\sum_{i=n}^{m-1}\mathsf{d}_{\sigma,\rho}(\zeta_i,\zeta_{i+1})\left[\prod_{\mathrm{j}=1}^{\mathrm{i}}\rho(\zeta_j,\zeta_m)\right]\sigma(\zeta_i,\zeta_{i+1}). \tag{4.11}$$

Note that the series

$$\sum_{n=1}^{\infty}\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})\left[\prod_{\mathrm{i}=1}^{n}\rho(\zeta_i,\zeta_m)\right]\sigma(\zeta_n,\zeta_{n+1}),$$

converges, by applying (4.8) and $(i),(ii)$, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})\left[\prod_{\mathrm{i}=1}^{\mathrm{i}}\rho(\zeta_i,\zeta_m)\right]\sigma(\zeta_n,\zeta_{n+1}) &\leq \sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{\mathfrak{p}}}}\left[\prod_{\mathrm{i}=1}^{\mathrm{i}}\rho(\zeta_i,\zeta_m)\right]\sigma(\zeta_n,\zeta_{n+1})\\ &< \frac{1}{\mathfrak{p}}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{\mathfrak{p}}}},\end{aligned}$$

which is convergent as $\frac{1}{\mathfrak{p}}>1$.

Suppose

$$\mathsf{R}=\sum_{n=1}^{\infty}\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})\left[\prod_{\mathrm{i}=1}^{\mathrm{j}}\rho(\zeta_i,\zeta_m)\right]\sigma(\zeta_j,\zeta_{j+1}),$$

$$\mathsf{R}_n = \sum_{j=1}^{n} \mathsf{d}_{\sigma,\rho}(\zeta_j, \zeta_{j+1}) \left[\prod_{\mathrm{i}=1}^{\mathrm{i}} \rho(\zeta_i, \zeta_m) \right] \sigma(\zeta_n, \zeta_{n+1}).$$

From above (4.11), can be written as,

$$\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_m) \leq \mathsf{R}_{\mathrm{m}-1} - \mathsf{R}_{\mathrm{n}-1}.$$

By taking limit as n, m approaches to infinity and using (4.8) we obtain

$$\lim_{n,m \to \infty} \mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_m) = 0.$$

Thus, $\{\zeta_n\}_{n \geq 0}$ is a Cauchy sequence in complete double controlled MS $(\mathsf{G}, \mathsf{d}_{\sigma,\rho})$, hence it converges to some $\zeta^* \in \mathsf{G}$.

To prove ζ^* is FP of $\Re_2$ assume on contrary that ζ^* does not belong to $[\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}$ (that is, $\mathsf{d}_{\sigma,\rho}(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}) > 0$), then there are $n_0 \in \mathbb{N}$ and a subsequence $\{\zeta_{n_k}\}$ of ζ_n such that $\mathsf{d}_{\sigma,\rho}(\zeta_{2n_k+1}, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}) > 0$, for all $n_k \geq n_0$. Since $\mathsf{d}_{\sigma,\rho}(\zeta_{2n_k+1}, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}) > 0$, we have

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{2n_k+1}, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}})) \leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{2n_k}, \zeta^*))]^{\mathsf{r}}. \tag{4.12}$$

Using the continuity of θ and by taking limit as $n \longrightarrow \infty$ in (4.12),

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}})) \leq 0,$$

which is a contradiction. Hence $\mathsf{d}_{\sigma,\rho}(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}) = 0$, and $\zeta^* \in [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}$.
By the same process $\zeta^* \in [\Re_1(\zeta^*)]_{\alpha_{\Re_1(\zeta^*)}}$.
Therefore $\zeta^* \in [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}} \cap [\Re_1(\zeta^*)]_{\alpha_{\Re_1(\zeta^*)}}$. □

Example 4.2.3. Let $\mathsf{G} = [0, 1]$. Define $\mathsf{d}_{\sigma,\rho} : \mathsf{G} \times \mathsf{G} \longrightarrow [0, \infty)$ by

$$\mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2) = |\zeta_1 - \zeta_2|^2,$$

Then $(\mathsf{d}_{\sigma,\rho}, \mathsf{G})$ is a complete double controlled MS, where $\sigma, \rho : \mathsf{G} \times \mathsf{G} \longrightarrow (0,1]$ is defined by

$$\sigma(\zeta_1, \zeta_2) = \begin{cases} 1, & \text{if } \zeta_1, \zeta_2 \in [0, 0.5), \\ \zeta_1 + \zeta_2 + 3, & \text{otherwise,} \end{cases}$$

and

$$\rho(\zeta_2, \zeta_3) = \begin{cases} 1, & \text{if } \zeta_2, \zeta_3 \in [0, 0.5), \\ 4 + \zeta_2 + \zeta_3, & \text{otherwise.} \end{cases}$$

Define $\Re_1, \Re_2 : \mathsf{G} \to \mathcal{F}(\mathsf{G})$ by,

$$\Re_1(\zeta_1)\Re_2 = \begin{cases} \alpha, & \text{if } 0 \leq \Re_2 \leq \frac{\zeta_1}{45}, \\ \frac{\alpha}{45}, & \text{if } \frac{\zeta_1}{45} \leq \Re_2 < \frac{\zeta_1}{30}, \\ \frac{\alpha}{30}, & \text{if } \frac{\zeta_1}{30} \leq \Re_2 < \frac{\zeta_1}{20}, \\ \frac{\alpha}{20}, & \text{if } \frac{\zeta_1}{15} < \Re_2 \leq 1, \end{cases}$$

and $\Re_2(\zeta_1)\Re_2 = \begin{cases} \alpha, & \text{if } 0 \leq \Re_2 \leq \frac{\zeta_1}{15}, \\ \frac{\alpha}{15}, & \text{if } \frac{\zeta_1}{15} \leq \Re_2 < \frac{\zeta_1}{10}, \\ \frac{\alpha}{10}, & \text{if } \frac{\zeta_1}{10} \leq \Re_2 < \frac{\zeta_1}{5}, \\ \frac{\alpha}{5}, & \text{if } \frac{\zeta_1}{5} < \Re_2 \leq 1. \end{cases}$

Now

$$[\Re_1(\zeta_1)]_{\alpha_{\Re_1}} = \left[0, \frac{\zeta_1}{45}\right],$$
$$[\Re_2(\zeta_1)]_{\alpha_{\Re_2}} = \left[0, \frac{\zeta_1}{15}\right].$$

Since,

$\mathrm{H}_{\sigma,\rho}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) = |\frac{\zeta_1}{45} - \frac{\zeta_2}{15}|^2 \geq 0$ for $\zeta_1 \neq \zeta_2$.

$$\begin{aligned}\mathrm{H}_{\sigma,\rho}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) &= |\frac{\zeta_1}{45} - \frac{\zeta_2}{15}|^2 \\ &\leq (\frac{1}{15})^2|\zeta_1 - \zeta_2|^2 \\ &\leq \frac{1}{15}|\zeta_1 - \zeta_2|^2\end{aligned}$$

for $\zeta_1 \neq \zeta_2$.

Hence,

$$\theta(\mathrm{H}_{\sigma,\rho}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}})) \leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2))]^{\mathsf{r}}, \text{ where } \theta(t) = e^{\sqrt{\mathsf{t}e^{\mathsf{t}}}}. \tag{4.13}$$

$$\theta(\frac{1}{15}|\zeta_1 - \zeta_2|^2) \leq [\theta(|\zeta_1 - \zeta_2|^2)]^{\mathsf{r}}. \tag{4.14}$$

So all the axioms of Theorem 4.2.2 with $\mathsf{r} = \frac{1}{2}$ are satisfied and 0 is a common FP.

If $\sigma(\zeta_1, \zeta_2) = \rho(\zeta_1, \zeta_2)$, in Theorem 4.2.2, by modifying the conditions (i) and (ii) the following corollary is obtained in case of controlled MS.

Corollary 4.2.4. Assume that $(\mathsf{G}, \mathsf{d}_\sigma)$ is a complete controlled MS, and let $\Re_1, \Re_2$ be the pair of FM satisfying the Θ-contraction. Then $\exists\ \zeta^* \in \mathsf{G}$, such that

$$\zeta^* \in \{[\Re_1(\zeta^*)]_{\alpha_{\Re_1(\zeta^*)}} \cap [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}\}.$$

If $\sigma(\zeta_1, \zeta_2) = \rho(\zeta_2, \zeta_3) = \rho(\zeta_1, \zeta_3)$, in Theorem 4.2.2, by modifying the conditions (i) and (ii) the following corollary is obtained in case of extended b-MS.

Corollary 4.2.5. Assume that $(\mathsf{G}, \mathsf{d}_\sigma)$ is a complete extended b-MS and let $\Re_1, \Re_2$ be the pair of FM satisfying the Θ-contraction. Then $\exists\ \zeta^* \in \mathsf{G}$, such that

$$\zeta^* \in \{[\Re_1(\zeta^*)]_{\alpha_{\Re_1(\zeta^*)}} \cap [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}\}.$$

If $\sigma(\zeta_1, \zeta_2) = \rho(\zeta_2, \zeta_3) = s$, (where $s \geq 1$) in Theorem 4.2.2, by modifying the conditions (i) and (ii) the following corollary is obtained in case of b-MS.

Corollary 4.2.6. Assume that $(\mathsf{G}, \mathsf{d}_\sigma)$ is a complete b-MS, and let $\Re_1, \Re_2$ be the pair of FM satisfying Θ-contraction. Then $\Re_1$ and $\Re_2$ have a common FP.

If $\sigma(\zeta_1, \zeta_2) = \rho(\zeta_2, \zeta_3) = 1$, in Theorem 4.2.2, by modifying the conditions (i) and (ii) the following corollary is obtained in case of MS.

Corollary 4.2.7. Assume that $(\mathsf{G}, \mathsf{d}_\sigma)$ is a complete MS, and let $\Re_1, \Re_2$ be the pair of FM satisfying Θ-contraction. Then $\Re_1$ and $\Re_2$ have a common FP.

If $\Re_1 = \Re_2$ in Definition 4.2.1, we obtain.

Corollary 4.2.8. Assume that $(\mathsf{G}, \mathsf{d}_{\sigma,\rho})$ is a complete double controlled MS, and let $\Re_1$ be the FM from G into $\mathcal{F}(\mathsf{G})$. Assume that for each $\zeta_1 \in \mathsf{G}$, $\exists\, \alpha_{\Re_1(\zeta_1)}, \alpha_{\Re_1(\zeta_2)} \in (0, 1]$ such that $[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}$ and $[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}} \in \mathfrak{CB}(\mathsf{G})$ are non empty sets. If $\theta \in \Theta$ such that $\mathrm{H}_{\sigma,\rho}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}) > 0$ implies,

$$[\theta(\mathrm{H}_{\sigma,\rho}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}))] \leq [\mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2))]^{\mathsf{r}}.$$

Then, $\Re_1$ has a FP.

4.3 Θ-Fuzzy Almost Generalized Double Controlled Contraction Mapping

This section deals with common α-fuzzy FP result in the context of double controlled MS. Motivated by the idea of a almost contraction for FM on MS, we have introduced the notion of $\Theta - \mathtt{FAGDCCM}$ as follows:

Definition 4.3.1. Assume that $(\mathsf{G}, \mathsf{d}_{\sigma,\rho})$ is a double controlled MS, then the pair $\Re_1, \Re_2 : \mathsf{G} \to \mathcal{F}(\mathsf{G})$ is known as $\Theta - \mathtt{FAGDCCM}$ if for all $\zeta_1, \zeta_2 \in \mathsf{G}$, $\mathfrak{L} \geq 0$ and $\theta \in \Theta$, $\mathrm{H}_{\sigma,\rho}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) > 0$ implies,

$$\theta(\mathrm{H}_{\sigma,\rho}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}})) \leq [\theta(\mathtt{M}(\zeta_1, \zeta_2))]^{\mathsf{r}} + \mathfrak{L}\Xi(\zeta_1, \zeta_2),$$

with

$$\mathtt{M}(\zeta_1,\zeta_2) = \max\Bigg\{ \mathsf{d}_{\sigma,\rho}(\zeta_2, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}), \mathsf{d}_{\sigma,\rho}(\zeta_1, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}),$$
$$\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2), \frac{\{\mathsf{d}_{\sigma,\rho}(\zeta_2, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) \times \mathsf{d}_{\sigma,\rho}(\zeta_1, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}})\}}{1+\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)},$$
$$\mathsf{d}_{\sigma,\rho}(\zeta_2, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}) \Bigg\},$$

and

$$\Xi(\zeta_1,\zeta_2) = \min\Bigg\{ \mathsf{d}_{\sigma,\rho}(\zeta_1, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}), \mathsf{d}_{\sigma,\rho}(\zeta_2, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}),$$
$$\mathsf{d}_{\sigma,\rho}(\zeta_1, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}), \mathsf{d}_{\sigma,\rho}(\zeta_2, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}) \Bigg\},$$

also

$[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}$, $[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}} \in \mathfrak{CB}(\mathsf{G})$

are non empty sets, $\alpha_{\Re_1(\zeta_1)}, \alpha_{\Re_2(\zeta_1)} \in (0,1]$

and $\mathsf{r} \in (0,1)$.

Theorem 4.3.2. Assume that $(\mathsf{G}, \mathsf{d}_{\sigma,\rho})$ is a complete double controlled MS, and $\Re_1, \Re_2 : \mathsf{G} \to \mathcal{F}(\mathsf{G})$ be the pair of $\Theta - \mathtt{FAGDCCM}$ satisfying the following:

(i): $\sup_{m\geq 0} \lim_{n\longrightarrow\infty} \sigma(\zeta_{n+1},\zeta_{n+2})\rho(\zeta_{n+1},\zeta_m) < \frac{1}{\mathfrak{p}}$.

(ii): Both $\lim_{n\longrightarrow\infty} \sigma(\zeta,\zeta_n)$ and $\lim_{n\longrightarrow\infty} \rho(\zeta_n,\zeta)$ exists, and are finite $\forall\ \zeta \in \mathsf{G}$ and the sequences $\{\zeta_{2n}\}$ and $\{\zeta_{2n+1}\}$ are defined as $\zeta_{2n} = {\Re_1}^n\zeta_0$ and $\zeta_{2n+1} = {\Re_2}^n\zeta_0$ for some $\zeta_0 \in \mathsf{G}$.

Then $\Re_1$ and $\Re_2$ have a common FP.

Proof. Let $\zeta_0 \in \mathsf{G}$. By hypothesis, $\exists$ $\alpha_{\Re_1(\zeta_0)} \in (0,1]$ such that such that $[\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}$ is a non empty element of $\mathfrak{CB}(\mathsf{G})$. Let $\zeta_1 \in [\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}$

also $\exists\ \alpha_{\Re_2(\zeta_1)} \in (0,1]$

such that

$[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}$ is a non empty element of $\mathfrak{CB}(\mathsf{G})$.

Since θ is strictly nondecreasing and by using Definition 4.1.1

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_1, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}) \leq \theta(\mathrm{H}_{\sigma,\rho}([\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}})). \tag{4.15}$$

Since,

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_1, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}})) = \inf_{\mathsf{a}\in[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}} \theta(\mathsf{d}_{\sigma,\rho}(\zeta_1, \mathsf{a})).$$

By hypothesis

$$\inf_{\mathsf{a}\in[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}} [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_1, \mathsf{a}))] \leq [\theta(\mathtt{M}(\zeta_0, \zeta_1))]^{\mathsf{r}} + \mathfrak{L}\Xi(\zeta_0, \zeta_1),$$

with

$$\begin{aligned}
&\mathtt{M}(\zeta_0, \zeta_1) \\
&= \max\Bigg\{\mathsf{d}_{\sigma,\rho}(\zeta_0, \zeta_1), \mathsf{d}_{\sigma,\rho}(\zeta_0, [\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}), \mathsf{d}_{\sigma,\rho}(\zeta_1, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}), \\
&\mathsf{d}_{\sigma,\rho}(\zeta_1, [\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}), \frac{\{\mathsf{d}_{\sigma,\rho}(\zeta_0, [\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}) \times \mathsf{d}_{\sigma,\rho}(\zeta_1, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}})\}}{1 + \mathsf{d}_{\sigma,\rho}(\zeta_0, \zeta_1)}\Bigg\},
\end{aligned}$$

and

$$\begin{aligned}
\Xi(\zeta_0, \zeta_1) = \min\Bigg\{&\mathsf{d}_{\sigma,\rho}(\zeta_0, [\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}), \mathsf{d}_{\sigma,\rho}(\zeta_1, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}), \\
&\mathsf{d}_{\sigma,\rho}(\zeta_0, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}), \mathsf{d}_{\sigma,\rho}(\zeta_1, [\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}})\Bigg\}.
\end{aligned}$$

Also $\exists\ \zeta_2 \in [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}$ such that

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2)) \leq [\theta(\mathtt{M}(\zeta_0, \zeta_1))]^{\mathsf{r}} + \mathfrak{L}\Xi(\zeta_0, \zeta_1). \tag{4.16}$$

Now,

$$\begin{aligned}
&\mathtt{M}(\zeta_0,\zeta_1)\\
&\leq \max\Bigg\{\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_1),\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1),\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1),\\
&\quad\frac{\{\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)\times\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1)\}}{1+\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1)},\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)\Bigg\}\\
&= \max\Bigg\{\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1),\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)\Bigg\},
\end{aligned}$$

and

$$\begin{aligned}
\Xi(\zeta_0,\zeta_1) &= \min\Bigg\{\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_1),\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1),\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_2),\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)\Bigg\},\\
&= \mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_1)\\
&= 0.
\end{aligned}$$

If we take

$$\max\Big\{\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1),\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)\Big\}=\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2).$$

Then (4.16) becomes,

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2))\leq[\theta(\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2))]^{\mathsf{r}},$$

a contradiction.

Thus,

$$\max\Big\{\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1),\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)\Big\}=\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1).$$

So (4.16) becomes,

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2))\leq[\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1))]^{\mathsf{r}}.$$

Now $\exists\ \alpha_{\Re_1(\zeta_2)}\in(0,1]$

such that

$[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}$ is a non empty element of $\mathfrak{CB}(\mathsf{G})$.

By Lemma 4.1.1 and contraction condition (4.3.1),

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_2,[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}))\leq\theta(\mathrm{H}_{\sigma,\rho}([\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}},[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}})).\tag{4.17}$$

Since,

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_2,[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}))=\inf_{\mathsf{e}\in[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}}\theta(\mathsf{d}_{\sigma,\rho}(\zeta_2,\mathsf{e})).$$

By hypothesis

$$\inf_{\mathsf{e}\in[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}}[\theta(\mathsf{d}_{\sigma,\rho}(\zeta_2,\mathsf{e}))]\leq[\theta(\mathtt{M}(\zeta_1,\zeta_2))]^{\mathsf{r}}+\mathfrak{L}\Xi(\zeta_1,\zeta_2),$$

with

$$\begin{aligned}&\mathtt{M}(\zeta_1,\zeta_2)\\&=\max\Bigg\{\mathsf{d}_{\sigma,\rho}(\zeta_2,[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}),\mathsf{d}_{\sigma,\rho}(\zeta_1,[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}),\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2),\\&\qquad\frac{\{\mathsf{d}_{\sigma,\rho}(\zeta_2,[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}})\times\mathsf{d}_{\sigma,\rho}(\zeta_1,[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}})\}}{1+\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)},\\&\qquad\mathsf{d}_{\sigma,\rho}(\zeta_2,[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}})\Bigg\},\end{aligned}$$

and

$$\begin{aligned}\Xi(\zeta_1,\zeta_2)=\min\Bigg\{&\mathsf{d}_{\sigma,\rho}(\zeta_2,[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}),\mathsf{d}_{\sigma,\rho}(\zeta_1,[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}),\\&\mathsf{d}_{\sigma,\rho}(\zeta_2,[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}),\mathsf{d}_{\sigma,\rho}(\zeta_1,[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}})\Bigg\}.\end{aligned}$$

Thus, $\exists\quad\zeta_3\in[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}$ such that,

$$[\theta(\mathsf{d}_{\sigma,\rho}(\zeta_2,\zeta_3))]\leq[\theta(\mathtt{M}(\zeta_1,\zeta_2))]^{\mathsf{r}}+\mathfrak{L}\Xi(\zeta_1,\zeta_2).\tag{4.18}$$

Now

$$\mathtt{M}(\zeta_1,\zeta_2) \leq \max\Bigg\{\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2),\mathsf{d}_{\sigma,\rho}(\zeta_2,\zeta_3),\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2),\mathsf{d}_{\sigma,\rho}(\zeta_2,\zeta_2),$$

$$\frac{\{\mathsf{d}_{\sigma,\rho}(\zeta_2,\zeta_3)\times\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)\}}{1+\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)}\Bigg\}.$$

$$= \max\Big\{\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2),\mathsf{d}_{\sigma,\rho}(\zeta_2,\zeta_3)\Big\},$$

and

$$\Xi(\zeta_1,\zeta_2) = \min\Big\{\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2),\mathsf{d}_{\sigma,\rho}(\zeta_2,\zeta_3),\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_3),\mathsf{d}_{\sigma,\rho}(\zeta_2,\zeta_2)\Big\},$$

$$=\mathsf{d}_{\sigma,\rho}(\zeta_2,\zeta_2)$$

$$=0.$$

If we take,

$$\max\Big\{\mathsf{d}_{\sigma,\rho}(\zeta_2,\zeta_3),\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)\Big\} = \mathsf{d}_{\sigma,\rho}(\zeta_2,\zeta_3).$$

Then (4.18) becomes,

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_2,\zeta_3)) \leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_2,\zeta_3))]^{\mathsf{r}},$$

a contradiction.

Thus,

$$\max\Big\{\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2),\mathsf{d}_{\sigma,\rho}(\zeta_2,\zeta_3)\Big\} = \mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2).$$

Hence

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_2,\zeta_3)) \leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2))]^{\mathsf{r}}.$$

Using this procedure, we generate a sequence $\{\zeta_n\} \in \mathsf{G}$ such that $\zeta_{2n+2} \in [\Re_2(\zeta_{2n+1})]_{\alpha_{\Re_2(\zeta_{2n+1})}}$ and $\zeta_{2n+1} \in [\Re_1(\zeta_{2n})]_{\alpha_{\Re_1(\zeta_{2n})}}$ satisfying

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{2n+1},\zeta_{2n+2})) \leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{2n},\zeta_{2n+1}))]^{\mathsf{r}}\ \forall\ n \in \mathbb{N}, \tag{4.19}$$

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{2n+2}, \zeta_{2n+3})) \leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{2n+1}, \zeta_{2n+2}))]^{\mathsf{r}} \ \forall \ n \in \mathbb{N}. \tag{4.20}$$

By combing (4.19) and (4.20)

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})) \leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{n-1}, \zeta_n))]^{\mathsf{r}} \text{ for all } n \in \mathbb{N},$$

which further implies

$$\begin{aligned}
\theta(\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})) &\leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{n-1}, \zeta_n))]^{\mathsf{r}} \\
&\leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{n-2}, \zeta_{n-1}))]^{\mathsf{r}^2} \\
&\leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{n-3}, \zeta_{n-2}))]^{\mathsf{r}^3} \\
&\vdots \\
&\leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0, \zeta_1))]^{\mathsf{r}^n}.
\end{aligned}$$

As $\theta(t) > 1$, we have

$$1 < \theta(\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})) \leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0, \zeta_1))]^{\mathsf{r}^n}.$$

By taking n tends to infinity we get

$$\lim_{n \longrightarrow \infty} \theta(\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})) = 1, \text{ as } \theta \in \Theta.$$

Thus, by Θ_2

$$\lim_{n \longrightarrow \infty} \mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1}) = 0^+.$$

In the view Θ_3, $\exists$ $\mathsf{p} \in (0, 1)$ and $\mathsf{M} \in (0, \infty]$ so that

$$\lim_{n \longrightarrow \infty} \frac{\theta(\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})) - 1}{\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})^{\mathsf{p}}} = \mathsf{M}.$$

Case 1: Let $0 < \mathsf{M} < \infty$ and $\frac{\mathsf{M}}{2} = \mathtt{C}$. Hence $\exists$ $n_0 \in \mathbb{N}$ such that for all $n > n_0$

$$\left| \frac{\theta(\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})) - 1}{\mathsf{d}_{\sigma,\rho}(\zeta_n, \zeta_{n+1})^{\mathsf{p}}} - \mathsf{M} \right| \leq \mathtt{C}.$$

that is

$$\frac{\theta(\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1}))-1}{\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})^{\mathsf{p}}} \geq \mathsf{M}-\mathsf{C}=\mathsf{C}.$$

Then,

$$n[\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})]^{\mathsf{p}} \leq \frac{n}{\mathsf{C}}([\theta(\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1}))]-1).$$

$$n[\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})]^{\mathsf{p}} \leq \frac{n}{\mathsf{C}}([\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1))]^{\mathsf{p}^n}-1).$$

By letting n tends to infinity and using (4.7) we obtain

$$\lim_{n\longrightarrow\infty} n[\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})]^{\mathsf{p}}=0.$$

Case 2: Let $\mathsf{M}=\infty$ and $\mathsf{C}>0$. Then $\exists\ n_1\in\mathbb{N}$ such that $\forall\ n>n_1$

$$\mathsf{C} \leq \frac{\theta(\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1}))-1}{\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})^{\mathsf{p}}}.$$

$$\frac{\theta(\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})))-1}{\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})^{\mathsf{p}}} \geq \mathsf{M}-\mathsf{C}=\mathsf{C}.$$

Then,

$$n[\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})]^{\mathsf{p}} \leq \frac{n}{\mathsf{C}}([\theta(\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1}))]-1).$$

$$n[\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})]^{\mathsf{p}} \leq \frac{n}{\mathsf{C}}([\theta(\mathsf{d}_{\sigma,\rho}(\zeta_0,\zeta_1))]^{\mathsf{p}^n}-1).$$

By taking limit as n approaches to infinity and using (4.7) we obtain

$$\lim_{n\longrightarrow\infty} n[\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})]^{\mathsf{p}}=0.$$

In both cases $\frac{1}{\mathsf{C}}>0$ we conclude that for any $\mathsf{M}\in(0,\infty]$ and $0<\mathsf{p}<1$ $\exists$ some $\mathsf{n}\in\mathbb{N}$, where $\mathsf{N}=\max\{n_0,n_1\}$. Therefore there is n_1 such that,

$$n[\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1})]^{\mathsf{p}} \leq 1\ \forall\ n>n_1).$$

This implies that

$$\mathsf{d}_{\sigma,\rho}(\zeta_n,\zeta_{n+1}) \leq \frac{1}{n^{\frac{1}{\mathsf{p}}}}. \tag{4.21}$$

Now proceeding as in of Theorem 4.2.2 we conclude that $\{\zeta_n\}$ is a Cauchy sequence in G. Thus, $\exists\ \zeta^*\in\mathsf{G}$ so that $\lim_{n\to\infty}\zeta_n\to\zeta^*$.

Next to show that ζ^* is a FP. We assert that $\{\zeta^*\} \in [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}$.
Assume that ζ^* does not belong to $[\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}$ (that is, $\mathsf{d}_{\sigma,\rho}(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}) > 0$), then $\exists$ $n_0 \in \mathbb{N}$ and a subsequence $\{\zeta_{n_k}\}$ of ζ_n so that

$$\mathsf{d}_{\sigma,\rho}(\zeta_{2n_k+1}, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}) > 0, \text{ for all } n_k \geq n_0.$$

Since $\mathsf{d}_{\sigma,\rho}(\zeta_{2n_k+1}, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}) > 0$, we have

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta_{2n_k+1}, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}})) \leq [\theta(\mathtt{M}(\zeta_{2n_k}, \zeta^*))]^{\mathsf{r}} + \mathfrak{L}\Xi(\zeta_{2n_k}, \zeta^*). \tag{4.22}$$

Now

$$\begin{aligned}\mathtt{M}(\zeta_{2n_k}, \zeta^*) = \max\Bigg\{&\mathsf{d}_{\sigma,\rho}(\zeta_{2n_k}, \zeta^*), \mathsf{d}_{\sigma,\rho}(\zeta_{2n_k}, [\Re_1(\zeta_{2n_k})]_{\alpha_{\Re_1(\zeta_{2n_k})}}),\\ &\mathsf{d}_{\sigma,\rho}(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}), \mathsf{d}_{\sigma,\rho}(\zeta^*, [\Re_1(\zeta_{2n_k})]_{\alpha_{\Re_1(\zeta_{2n_k})}}),\\ &\frac{\{\mathsf{d}_{\sigma,\rho}(\zeta_{2n_k}, [\Re_1(\zeta_{2n_k})]_{\alpha_{\Re_1(\zeta_{2n_k})}}) \times \mathsf{d}_{\sigma,\rho}(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}})\}}{1 + \mathsf{d}_{\sigma,\rho}(\zeta_{2n_k}, \zeta^*)}\Bigg\}.\\ \leq \max\Big\{&\mathsf{d}_{\sigma,\rho}(\zeta_{2n_k}, \zeta^*), \mathsf{d}_{\sigma,\rho}(\zeta_{2n_k}, \zeta_{2n_k+1}), \mathsf{d}_{\sigma,\rho}(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}),\\ &\mathsf{d}_{\sigma,\rho}(\zeta^*, \zeta_{2n_k+1}), \frac{\{\mathsf{d}_{\sigma,\rho}(\zeta_{2n_k}, \zeta_{2n_k+1}) \times \mathsf{d}_{\sigma,\rho}(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}})\}}{1 + \mathsf{d}_{\sigma,\rho}(\zeta_{2n_k}, \zeta^*)}\Big\},\end{aligned}$$

and

$$\begin{aligned}&\Xi(\zeta_{2n_k}, \zeta^*) =\\ &\min\Bigg\{\mathsf{d}_{\sigma,\rho}(\zeta_{2n_k}, [\Re_1(\zeta_{2n_k})]_{\alpha_{\Re_1(\zeta_{2n_k})}}), \mathsf{d}_{\sigma,\rho}(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}),\\ &\quad \mathsf{d}_{\sigma,\rho}(\zeta_{2n_k}, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}) + \mathsf{d}_{\sigma,\rho}(\zeta^*, [\Re_1(\zeta_{2n_k})]_{\alpha_{\Re_1(\zeta_{2n_k})}})\Bigg\}.\\ &\leq \min\Bigg\{\mathsf{d}_{\sigma,\rho}(\zeta_{2n_k}, \zeta_{2n_k+1}), \mathsf{d}_{\sigma,\rho}(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}),\\ &\quad \mathsf{d}_{\sigma,\rho}(\zeta_{2n_k}, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}) + \mathsf{d}_{\sigma,\rho}(\zeta^*, \zeta_{2n_k+1})\Bigg\}.\end{aligned}$$

By taking limit n tends to infinity in (4.22) we get,

$$\theta(\mathsf{d}_{\sigma,\rho}(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}})) \leq [\theta(\mathsf{d}_{\sigma,\rho}(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}))]^{\mathsf{r}},$$

which is again a contradiction.

Hence $\mathsf{d}_{\sigma,\rho}(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}) = 0$, and $\zeta^* \in [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}$.
By the above process $\zeta^* \in [\Re_1(\zeta^*)]_{\alpha_{\Re_1(\zeta^*)}}$. Therefore $\zeta^* \in [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}} \cap [\Re_1(\zeta^*)]_{\alpha_{\Re_1(\zeta^*)}}$. □

By taking $\mathfrak{L} = 0$ in Definition 4.3.1 we obtain.

Theorem 4.3.3. Assume that $(\mathsf{G}, \mathsf{d}_{\sigma,\rho})$ is a complete double controlled MS, and let $\Re_1, \Re_2$ be the pair of $\Theta-$FAGDCCM. Then $\exists\, \zeta^* \in \mathsf{G}$, such that $\zeta^* \in \{[\Re_1(\zeta^*)]_{\alpha_{\Re_1(\zeta^*)}} \cap [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}\}$, where $\mathtt{M}(\zeta_1, \zeta_2)$ is defined same in Definition 4.3.1.

If we take $\Re_1 = \Re_2$ in Definition 4.3.1, we have:

Corollary 4.3.4. Assume that $(\mathsf{G}, \mathsf{d}_{\sigma,\rho})$ is a complete double controlled MS, and let $\Re_1$ be the FM from G into $\mathcal{F}(\mathsf{G})$. Assume that for each $\zeta_1 \in \mathsf{G}$, $\exists\, \alpha_{\Re_1(\zeta_1)}, \alpha_{\Re_1(\zeta_2)} \in (0, 1]$ such that $[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}$ and $[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}$ are non empty and $\in \mathfrak{CB}(\mathsf{G})$. If $\exists \quad \theta \in \Theta$ and $\mathfrak{L} \geq 0$ such that $\mathrm{H}_{\sigma,\rho}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_1(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) > 0$ implies,

$$\theta(\mathrm{H}_{\sigma,\rho}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}})) \leq [\theta(\mathtt{M}(\zeta_1, \zeta_2))]^{\mathsf{r}} + \mathfrak{L}\Xi(\zeta_1, \zeta_2),$$

with

$$\begin{aligned} &\mathtt{M}(\zeta_1, \zeta_2) \\ &= \max\Bigg\{\mathsf{d}_{\sigma,\rho}(\zeta_1, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}), \mathsf{d}_{\sigma,\rho}(\zeta_2, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}), \mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2), \\ &\quad \frac{\{\mathsf{d}_{\sigma,\rho}(\zeta_2, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}) \times \mathsf{d}_{\sigma,\rho}(\zeta_1, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}})\}}{1 + \mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2)}, \\ &\quad \mathsf{d}_{\sigma,\rho}(\zeta_2, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}})\Bigg\}, \end{aligned}$$

and

$$\Xi(\zeta_1,\zeta_2)$$
$$= \min\Big\{ \mathsf{d}_{\sigma,\rho}(\zeta_2,[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}), \mathsf{d}_{\sigma,\rho}(\zeta_1,[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}), \mathsf{d}_{\sigma,\rho}(\zeta_2,[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}),$$
$$\mathsf{d}_{\sigma,\rho}(\zeta_1,[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}) \Big\}.$$

Then, $\exists \quad \zeta^* \in \mathsf{G}$, such that $\zeta^* \in \{[\Re_1(\zeta^*)]_{\alpha_{\Re_1(\zeta^*)}}\}$.

By taking $\mathfrak{L} = 0$ in Corollary 4.3.4 we get:

Corollary 4.3.5. Assume that $(\mathsf{G},\mathsf{d}_{\sigma,\rho})$ is a complete double controlled MS, and let $\Re_1 : \mathsf{G} \to \mathcal{F}(\mathsf{G})$ be the FM. If for each $\zeta_1 \in \mathsf{G}$, $\exists \quad \alpha_{\Re_1(\zeta_1)}, \alpha_{\Re_1(\zeta_2)} \in (0,1]$ such that $[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}$ is a non empty element of $\mathfrak{CB}(\mathsf{G})$. Assume $\exists\ \theta \in \Theta$ such that $\mathrm{H}_{\sigma,\rho}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}) > 0$ implies,

$$\theta(\mathrm{H}_{\sigma,\rho}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}})) \leq [\theta(\mathtt{M}(\zeta_1,\zeta_2))]^{\mathsf{r}},$$

where $\mathtt{M}(\zeta_1,\zeta_2)$ is defined same as in Corollary 4.3.4 and $\mathfrak{L} \geq 0$.
Then $\exists \quad \zeta^* \in \mathsf{G}$, such that $\zeta^* \in \{[\Re_1(\zeta^*)]_{\alpha_{\Re_1(\zeta^*)}}\}$.

4.4 Some Consequences

This section describes a few consequences of our results on MMs. In mathematics, a MMs, is a mapping that associates each element of a given set with a non-empty set of values.

Corollary 4.4.1. Assume that $(\mathsf{G},\mathsf{d}_{\sigma,\rho})$ is a complete double controlled MS, and let $\mathcal{K},\mathcal{J} : \mathsf{G} \to \mathfrak{CB}(\mathsf{G})$ be the pair of multi-valued almost Θ-contraction mappings i.e for each $\zeta_1,\zeta_2 \in \mathsf{G}$, $\exists \quad \mathfrak{L} \geq 0$ and $\theta \in \Theta$ such that $\mathrm{H}_{\sigma,\rho}(\mathcal{K}(\zeta_1),\mathcal{J}(\zeta_2)) > 0$,

$$\Rightarrow \theta(\mathrm{H}_{\sigma,\rho}(\mathcal{K}(\zeta_1),\mathcal{J}(\zeta_2))) \leq [\theta(\mathtt{M}(\zeta_1,\zeta_2))]^{\mathsf{r}} + \mathfrak{L}\Xi(\zeta_1,\zeta_2),$$

with

$$\begin{aligned}&\mathtt{M}(\zeta_1,\zeta_2)\\&=\max\Bigg\{\mathsf{d}_{\sigma,\rho}(\zeta_2,\mathcal{J}(\zeta_2),\mathsf{d}_{\sigma,\rho}(\zeta_2,\mathcal{K}(\zeta_1)),\mathsf{d}_{\sigma,\rho}(\zeta_1,\mathcal{K}(\zeta_1),\\&\qquad\frac{\{\mathsf{d}_{\sigma,\rho}(\zeta_2,\mathcal{J}(\zeta_2)\times\mathsf{d}_{\sigma,\rho}(\zeta_1,\mathcal{K}(\zeta_1)\}}{\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)+1},\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)\Bigg\},\end{aligned}$$

and

$$\begin{aligned}&\Xi(\zeta_1,\zeta_2)\\&=\min\Bigg\{\mathsf{d}_{\sigma,\rho}(\zeta_1,\mathcal{K}(\zeta_1)),\mathsf{d}_{\sigma,\rho}(\zeta_2,\mathcal{J}(\zeta_2)),\mathsf{d}_{\sigma,\rho}(\zeta_1,\mathcal{J}(\zeta_2)),\mathsf{d}_{\sigma,\rho}(\zeta_2,\mathcal{K}(\zeta_1))\Bigg\}.\end{aligned}$$

Then $\mathcal{K}$ and $\mathcal{J}$ has a common FP.

Proof. Consider a mapping $\alpha:\mathsf{G}\longrightarrow(0,1]$ and the pair of mappings $\Re_1,\Re_2:\mathsf{G}\to\mathcal{F}(\mathsf{G})$ by

$$\Re_1(\zeta_1)(\Re_2)=\begin{cases}\alpha, & \text{if } \Re_2\in\mathcal{K}(\zeta_1),\\ 0, & \text{if } \Re_2\notin\mathcal{K}(\zeta_1),\end{cases}$$

and

$$\Re_2(\zeta_1)(\Re_2)=\begin{cases}\alpha, & \text{if } \Re_2\in\mathcal{J}(\zeta_1),\\ 0, & \text{if } \Re_2\notin\mathcal{J}(\zeta_1).\end{cases}$$

Then we have the following:

$$\begin{aligned}[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}&=\Big\{\Re_2:\Re_1(\zeta_1)(\Re_2)\geq\alpha\Big\}=\mathcal{K}(\zeta_1).\\ [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}&=\Big\{\Re_2:\Re_2(\zeta_1)(\Re_2)\geq\alpha\Big\}=\mathcal{J}(\zeta_1).\end{aligned}$$

By Theorem 4.3.2 $\exists$ a FP $\zeta^*\in\{[\Re_1(\zeta^*)]_{\alpha_{\Re_1(\zeta^*)}}\cap[\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}\}=\mathcal{K}_{\zeta^*}\cap\mathcal{J}_{\zeta^*}$. □

If we take $\mathfrak{L}=0$ in Corollary 4.4.1 we get:

Corollary 4.4.2. Suppose that $(\mathsf{G},\mathsf{d}_{\sigma,\rho})$ be a complete double controlled MS, and let $\mathcal{K},\mathcal{J}:\mathsf{G}\to\mathfrak{CB}(\mathsf{G})$ be the pair of MMs i.e for each $\zeta_1,\zeta_2\in\mathsf{G}$, $\exists$ $\mathfrak{L}\geq 0$ and

$\theta \in \Theta$ such that $\mathrm{H}_{\sigma,\rho}(\mathcal{K}(\zeta_1), \mathcal{J}(\zeta_2)) > 0$,

$$\Rightarrow \theta(\mathrm{H}_{\sigma,\rho}(\mathcal{K}(\zeta_1), \mathcal{J}(\zeta_2))) \leq [\theta(\mathtt{M}(\zeta_1, \zeta_2))]^{\mathsf{r}},$$

where $\mathtt{M}(\zeta_1, \zeta_2)$ is defined same as in Corollary 4.4.1.

Then, $\mathcal{K}$ and $\mathcal{J}$ has a common FP.

By taking $\mathcal{K} = \mathcal{J}$ in Corollary 4.4.1 we obtain:

Corollary 4.4.3. Consider a complete double controlled MS, $(\mathsf{G}, \mathsf{d}_{\sigma,\rho})$ and let $\mathcal{K} : \mathsf{G} \to \mathfrak{CB}(\mathsf{G})$ be the MMs i.e for each $\zeta_1, \zeta_2 \in \mathsf{G}$, $\exists \quad \mathfrak{L} \geq 0$ and $\theta \in \Theta$ such that $\mathrm{H}_{\sigma,\rho}(\mathcal{K}(\zeta_1), \mathcal{J}(\zeta_2)) > 0$,

$$\Rightarrow \theta(\mathrm{H}_{\sigma,\rho}(\mathcal{K}(\zeta_1), \mathcal{K}(\zeta_2))) \leq [\theta(\mathtt{M}(\zeta_1, \zeta_2))]^{\mathsf{r}} + \mathfrak{L}\Xi(\zeta_1, \zeta_2),$$

with

$$\begin{aligned}
&\mathtt{M}(\zeta_1, \zeta_2) \\
&= \max\Bigg\{\mathsf{d}_{\sigma,\rho}(\zeta_2, \mathcal{K}(\zeta_2), \mathsf{d}_{\sigma,\rho}(\zeta_2, \mathcal{K}(\zeta_1)), \mathsf{d}_{\sigma,\rho}(\zeta_1, \mathcal{K}(\zeta_1), \mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2), \\
&\frac{\{\mathsf{d}_{\sigma,\rho}(\zeta_2, \mathcal{K}(\zeta_2) \times \mathsf{d}_{\sigma,\rho}(\zeta_1, \mathcal{K}(\zeta_1)\}}{1 + \mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2)}\Bigg\},
\end{aligned}$$

and

$$\begin{aligned}
&\Xi(\zeta_1, \zeta_2) \\
&= \min\Big\{\mathsf{d}_{\sigma,\rho}(\zeta_1, \mathcal{K}(\zeta_2)), \mathsf{d}_{\sigma,\rho}(\zeta_2, \mathcal{K}(\zeta_1)), \mathsf{d}_{\sigma,\rho}(\zeta_1, \mathcal{K}(\zeta_1)), \mathsf{d}_{\sigma,\rho}(\zeta_2, \mathcal{K}(\zeta_2))\Big\}.
\end{aligned}$$

Then there is a FP of $\mathcal{K}$.

By taking $\mathfrak{L} = 0$ in Corollary 4.4.3 we get:

Corollary 4.4.4. Assume that $(\mathsf{G}, \mathsf{d}_{\sigma,\rho})$ is a complete double controlled MS, and let $\mathcal{K} : \mathsf{G} \to \mathfrak{CB}(\mathsf{G})$ be the MMs i.e for each $\zeta_1, \zeta_2 \in \mathsf{G}$, $\exists \quad \mathfrak{L} \geq 0$ and $\theta \in \Theta$ such that $\mathrm{H}_{\sigma,\rho}(\mathcal{K}(\zeta_1), \mathcal{J}(\zeta_2)) > 0$,

$$\Rightarrow \theta(\mathrm{H}_{\sigma,\rho}(\mathcal{K}(\zeta_1), \mathcal{K}(\zeta_2))) \leq [\theta(\mathtt{M}(\zeta_1, \zeta_2))]^{\mathsf{r}},$$

where $\mathtt{M}(\zeta_1, \zeta_2)$ is defined same as in Corollary 4.4.3. Then, $\mathcal{K}$ has a FP.

Example 4.4.5. Let $\mathsf{G} = [0, 1]$. Define $\mathsf{d}_{\sigma,\rho} : \mathsf{G} \times \mathsf{G} \longrightarrow [0, \infty)$ by

$$\mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2) = |\zeta_1 - \zeta_2|^2.$$

Then $(\mathsf{d}_{\sigma,\rho}, \mathsf{G})$ is a complete double controlled MS, where $\sigma, \rho : \mathsf{G} \times \mathsf{G} \longrightarrow (0, 1]$ be defined by$\sigma(\zeta_1, \zeta_2) = \begin{cases} 1, & \text{if } \zeta_1, \zeta_2 \in [0, 0.5), \\ \zeta_1 + \zeta_2 + 2, & \text{otherwise,} \end{cases}$

and $\rho(\zeta_2, \zeta_3) = \begin{cases} 1, & \text{if } \zeta_2, \zeta_3 \in [0, 0.5), \\ \zeta_2 + \zeta_3 + 3, & \text{otherwise.} \end{cases}$

Define $\Re_1, \Re_2 : \mathsf{G} \to \mathcal{F}(\mathsf{G})$ by,

$$\Re_1(\zeta_1)\Re_2 = \begin{cases} \alpha, & \text{if } 0 \leq \Re_2 \leq \frac{\zeta_1}{45}, \\ \frac{\alpha}{3}, & \text{if } \frac{\zeta_1}{45} \leq \Re_2 \leq \frac{\zeta_1}{30}, \\ \frac{\alpha}{6}, & \text{if } \frac{\zeta_1}{30} \leq \Re_2 \leq \frac{\zeta_1}{20}, \\ \frac{\alpha}{9}, & \text{if } \frac{\zeta_1}{15} \leq \Re_2 \leq 1, \end{cases}$$

and $\Re_2(\zeta_1)\Re_2 = \begin{cases} \alpha, & \text{if } 0 \leq \Re_2 \leq \frac{\zeta_1}{15}, \\ \frac{\alpha}{9}, & \text{if } \frac{\zeta_1}{15} \leq \Re_2 \leq \frac{\zeta_1}{10}, \\ \frac{\alpha}{12}, & \text{if } \frac{\zeta_1}{10} \leq \Re_2 \leq \frac{\zeta_1}{5}, \\ \frac{\alpha}{15}, & \text{if } \frac{\zeta_1}{5} \leq \Re_2 \leq 1. \end{cases}$

Next

$$[\Re_1(\zeta_1)]_\alpha = \Big[0, \frac{\zeta_1}{45}\Big],$$
$$[\Re_2(\zeta_1)]_\alpha = \Big[0, \frac{\zeta_1}{15}\Big].$$

Since,

$$\mathrm{H}_{\sigma,\rho}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) = |\frac{\zeta_1}{45} - \frac{\zeta_2}{15}|^2 > 0$$

for $\zeta_1 \neq \zeta_2$, and for $\zeta_1 \neq \zeta_2$ we have

$$\begin{aligned}\mathrm{H}_{\sigma,\rho}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) &= |\frac{\zeta_1}{45} - \frac{\zeta_2}{15}|^2 \\ &\leq (\frac{1}{15})^2|\zeta_1 - \zeta_2|^2 \\ &\leq \frac{1}{15}|\zeta_1 - \zeta_2|^2.\end{aligned}$$

Now

$$\mathtt{M}(\zeta_1, \zeta_2)$$

$$= \max\Bigg\{\mathsf{d}_{\sigma,\rho}(\zeta_2, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}), \mathsf{d}_{\sigma,\rho}(\zeta_1, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}), \mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2),$$

$$\frac{\{\mathsf{d}_{\sigma,\rho}(\zeta_2, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) \times \mathsf{d}_{\sigma,\rho}(\zeta_1, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}})\}}{1 + \mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2)},$$

$$\mathsf{d}_{\sigma,\rho}(\zeta_2, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}})\Bigg\},$$

$$\Rightarrow \mathtt{M}(\zeta_1, \zeta_2)$$

$$= \max\Bigg\{|\zeta_1 - \zeta_2|^2, |\zeta_1 - \frac{\zeta_1}{45}|^2, |\zeta_2 - \frac{\zeta_2}{15}|^2, |\zeta_2 - \frac{\zeta_1}{45}|^2$$

$$\frac{\{|\zeta_1 - \frac{\zeta_1}{45}|^2 \times |\zeta_2 - \frac{\zeta_2}{15}|^2\}}{1 + |\zeta_1 - \zeta_2|^2}\Bigg\},$$

$$\leq |\zeta_1 - \zeta_2|^2,$$

and

$$\Xi(\zeta_1,\zeta_2)$$

$$= \min\Big\{\mathsf{d}_{\sigma,\rho}(\zeta_1,[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}),\mathsf{d}_{\sigma,\rho}(\zeta_2,[\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}),$$

$$\mathsf{d}_{\sigma,\rho}(\zeta_1,[\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}),\mathsf{d}_{\sigma,\rho}(\zeta_2,[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}})\Big\},$$

$$\Rightarrow \Xi(\zeta_1,\zeta_2)$$

$$= \min\Big\{|\zeta_1-\frac{\zeta_1}{45}|^2,|\zeta_2-\frac{\zeta_2}{15}|^2,|\zeta_1-\frac{\zeta_2}{15}|^2,|\zeta_2-\frac{\zeta_1}{45}|^2)\Big\}$$
$$\leq (\frac{1}{45})^2|\zeta_1-\zeta_2|^2$$
$$\leq \frac{1}{45}|\zeta_1-\zeta_2|^2.$$

Hence, $\mathrm{H}_{\sigma,\rho}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}},[\Re_1(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) > 0$ implies,

$$\theta(\mathrm{H}_{\sigma,\rho}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}},[\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}})) \leq [\theta(\mathtt{M}(\zeta_1,\zeta_2))]^{\mathsf{r}} + \mathfrak{L}\Xi(\zeta_1,\zeta_2),$$

where $\theta(\mathtt{t}) = e^{\sqrt{\mathtt{t}e^{\mathtt{t}}}}$ and $\mathfrak{L} = 1$.

So all the axioms of Theorem 4.3.2 with $\mathfrak{L} = 1$ is satisfied and $0 \in [\Re_2(0)]_{\alpha_{\Re_2}} \cap [\Re_1(0)]_{\alpha_{\Re_1}}$.

4.5 Application

This section provides an application of our results for the existence of a solution to a second-order non-linear boundary value problem. Consider the following second-order non-linear boundary value problem:

$$\begin{cases} \zeta_1''(\mathfrak{a}) = \mathtt{K}(\mathfrak{a},\zeta_1(\mathfrak{a}),\zeta_1'(\mathfrak{a})), & \mathfrak{a}\in[0,a], a>0 \\ \zeta_1(\mathfrak{a}_1) = \mathsf{f}_1, & \mathfrak{a}_1\in[0,a] \\ \zeta_1(\mathfrak{a}_2) = \mathsf{f}_2, & \mathfrak{a}_2\in[0,a], \end{cases}$$

where $\mathtt{K} : [0, a] \times \mathfrak{D}(\mathsf{G}) \times \mathfrak{D}(\mathsf{G}) \longrightarrow \mathfrak{D}(\mathsf{G})$ is a continuous function and $\mathfrak{D}(\mathsf{G})$ represent the collection of all approximate quantities in G.
$\zeta_1(\mathfrak{a}) = \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a}, \varrho)\mathtt{K}(\mathfrak{a}, \zeta_1(\varrho), \zeta_1'(\varrho))\, d\varrho \ + \Pi(\mathfrak{a}), \mathfrak{a} \in [0, a]$,
where Green's function $\mathfrak{J}$ is given by

$$\mathfrak{J}(\mathfrak{a}, \varrho) = \begin{cases} \dfrac{(\mathfrak{a}_2 - \mathfrak{a})(\varrho - \mathfrak{a})}{\mathfrak{a}_2 - \mathfrak{a}_1}, & \text{if } \ \mathfrak{a}_1 \le \varrho \le \mathfrak{a} \le \mathfrak{a}_2, \\ \dfrac{(\mathfrak{a}_2 - \varrho)(\mathfrak{a} - \mathfrak{a}_1)}{\mathfrak{a}_2 - \mathfrak{a}_1}, & \text{if } \ \mathfrak{a}_1 \le \mathfrak{a} \le \varrho \le \mathfrak{a}_2. \end{cases}$$

and $\Pi(\mathfrak{a})$ satisfies $\Pi''(\mathfrak{a}) = 0, \Pi(\mathfrak{a}_1) = \mathsf{f}_1$, and $\Pi(\mathfrak{a}_2) = \mathsf{f}_2$. Let us recall some properties of $\mathfrak{J}(\mathfrak{a}, \varrho)$. Particularly,

$$\begin{aligned} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |\mathfrak{J}(\mathfrak{a}, \varrho)| d\varrho \ &\le \frac{(\mathfrak{a}_2 - \mathfrak{a}_1)^2}{8}, \\ \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |\mathfrak{J}_{\mathfrak{a}}(\mathfrak{a}, \varrho)| d\varrho \ &\le \frac{(\mathfrak{a}_2 - \mathfrak{a}_1)}{2}. \end{aligned}$$

We aim to find the existence of a common FP of a pair of operators given as,

$$\Re_1(\zeta_1)(\mathfrak{a}) = \Pi(\mathfrak{a}) + \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a}, \varrho)\mathtt{K}_1(\mathfrak{a}, \zeta_1(\varrho), \zeta_1'(\varrho))\, d\varrho \ , \mathfrak{a} \in [0, a],$$

$$\Re_2(\zeta_2)(\mathfrak{a}) = \Pi(\mathfrak{a}) + \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a}, \varrho)\mathtt{K}_2(\mathfrak{a}, \zeta_2(\varrho), \zeta_2'(\varrho))\, d\varrho \ , \mathfrak{a} \in [0, a].$$

Where $\mathtt{K}_1, \mathtt{K}_2 \in \mathsf{C}([0, a] \times \mathfrak{D}(\mathsf{G}) \times \mathfrak{D}(\mathsf{G}), \mathfrak{D}(\mathsf{G}))$, $\zeta_1 \in \mathsf{C}([0, a], \mathfrak{D}(\mathsf{G}))$ and $\Pi \in \mathsf{C}^1([0, a], \mathfrak{D}(\mathsf{G}))$.

Theorem 4.5.1. Suppose that

(D_1): $\mathtt{K}_1, \mathtt{K}_2 : [0, a] \times \mathfrak{D}(\mathsf{G}) \times \mathfrak{D}(\mathsf{G}) \longrightarrow \mathfrak{D}(\mathsf{G})$ are increasing mappings in their second and third variables,

(D_2): There is $\zeta_0 \in \mathsf{C}^1([0, a], \mathfrak{D}(\mathsf{G}))$ such that $\mathfrak{a} \in [0, a]$, we have

$$\Re_1(\zeta_0)(\mathfrak{a}) = \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a}, \varrho)\mathtt{K}_1(\mathfrak{a}, \zeta_0(\varrho), \zeta_0'(\varrho))\, d\varrho \ + \Pi(\mathfrak{a}), \mathfrak{a} \in [0, a].$$

(D_3): There is $\mathfrak{a} \in [0, a]$, we have

$$\begin{aligned}
&|\mathtt{K}_1(\mathfrak{a}, \zeta_1(\varrho), \zeta_1'(\varrho)) \; - \mathtt{K}_2(\mathfrak{a}, \zeta_2(\varrho), \zeta_2'(\varrho)) \; | \\
&= |\zeta_1''(\varrho) - \zeta_2''(\varrho)| \\
&\leq \frac{1}{\mathfrak{p}}|\zeta_1(\varrho) - \zeta_2(\varrho)| + \frac{1}{\mathfrak{p}}|\zeta_1'(\varrho) - \zeta_2'(\varrho)|,
\end{aligned}$$

for all $\zeta_1, \zeta_2 \in \mathsf{C}([0, a], \mathfrak{D}(\mathsf{G}))$ with $\mathtt{K}_1(.,.,.,) \neq \mathtt{K}_2(.,.,.,)$.

(D_4): For $\lim_{n \longrightarrow \infty} \sigma(\zeta_1, \zeta_2) < \frac{1}{\mathfrak{p}} = 1.5$, $\lim_{n \longrightarrow \infty} \rho(\zeta_2, \zeta_3) < \frac{1}{\mathfrak{p}} = 1.5$ and $\forall \; \mathfrak{a}_1, \mathfrak{a}_2 \in [0, a]$, we have,

$$\begin{aligned}
\frac{(\mathfrak{a}_2 - \mathfrak{a}_1)^2}{8\mathfrak{p}} + \frac{(\mathfrak{a}_2 - \mathfrak{a}_1)}{2\mathfrak{p}} &< \frac{1.5(\mathfrak{a}_2 - \mathfrak{a}_1)^2}{8} + \frac{1.5(\mathfrak{a}_2 - \mathfrak{a}_1)}{2} \\
&< \frac{9}{10}
\end{aligned}$$

(D_5): $\zeta_1, \zeta_2 \in \mathsf{C}^1([0, a], \mathfrak{D}(\mathsf{G}))$ is comparable, then every $i \in (\Re_1(\zeta_1))_1$ and $j \in (\Re_1(\zeta_2))_1$ are comparable.

Then, the pair of integral equations,

$$\begin{aligned}
\Re_1(\zeta_1)(\mathfrak{a}) &= \Pi(\mathfrak{a}) + \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a}, \varrho)\mathtt{K}_1(\mathfrak{a}, \zeta_1(\varrho), \zeta_1'(\varrho)) \, d\varrho, \mathfrak{a} \in [0, a], \\
\Re_2(\zeta_2)(\mathfrak{a}) &= \Pi(\mathfrak{a}) + \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a}, \varrho)\mathtt{K}_2(\mathfrak{a}, \zeta_2(\varrho)), \zeta_2'(\varrho)) \, d\varrho, \mathfrak{a} \in [0, a],
\end{aligned}$$

has a common solution in $\mathsf{C}^1([\mathfrak{a}_1, \mathfrak{a}_2], \mathfrak{D}(\mathsf{G}))$.

Proof. Consider $\mathsf{C} = \mathsf{C}^1([\mathfrak{a}_1, \mathfrak{a}_2], \mathfrak{D}(\mathsf{G}))$ with double controlled MS.

$$\mathsf{d}_{\sigma,\rho}(\zeta_1, \zeta_2) = \max_{\varrho \in [0,a]} |\zeta_1(\varrho) - \zeta_2(\varrho)|^2.$$

Note that $\mathsf{d}_{\sigma,\rho}$ is assumed to be a complete double controlled MS. Let $\Re_1, \Re_2 : \mathsf{C} \longrightarrow \mathsf{C}$ be two integral operators as defined above. Clearly, $\Re_1$ and $\Re_2$ are well defined. Since $\mathtt{K}_1$ and $\mathtt{K}_2$ with Π are continuous functions. Now ζ^* is a solution if and only if ζ^* is a common FP for the pair of FM $(\Re_1, \Re_2)$. By (D_1), $\Re_1, \Re_2$ is increasing. Next for all $\zeta_1, \zeta_2 \in \mathsf{G}$ with $\mathtt{K}_1(.,.,.,) \neq \mathtt{K}_2(.,.,.,)$, by (D_3), we have,

$$\begin{aligned}
&|\Re_1(\zeta_1)(\mathfrak{a}) - \Re_2(\zeta_2)(\mathfrak{a})| \\
&= \Bigg|\Big(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a},\varrho)\mathtt{K}_1(\mathfrak{a},\zeta_1(\varrho),\zeta_1'(\varrho))\,d\varrho \ + \Pi(\mathfrak{a})\Big) - \\
&\Big(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a},\varrho)\mathtt{K}_2(\mathfrak{a},\zeta_2(\varrho),\zeta_2'(\varrho))\ d\varrho \ + \Pi(\mathfrak{a})\Big)\Bigg|, \qquad \mathfrak{a} \in [0,a], \\
&= \left|\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a},\varrho)(\mathtt{K}_1(\mathfrak{a},\zeta_1(\varrho),\zeta_1'(\varrho)) - \mathtt{K}_2(\mathfrak{a},\zeta_2(\varrho),\zeta_2'(\varrho)))d\varrho\right|, \qquad \mathfrak{a} \in [0,a], \qquad (4.23) \\
&\leq \left|\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a},\varrho)||\mathtt{K}_1(\mathfrak{a},\zeta_1(\varrho),\zeta_1'(\varrho)) - \mathtt{K}_2(\mathfrak{a},\zeta_2(\varrho),\zeta_2'(\varrho)))\right| d\varrho, \qquad \mathfrak{a} \in [0,a], \\
&\leq \max_{\varrho\in[0,a]} |\zeta_1(\varrho) - \zeta_2(\varrho)|^2 \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |\mathfrak{J}(\mathfrak{a},\varrho)|d\varrho, \\
&\leq \mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)\frac{1.5(\mathfrak{a}_2-\mathfrak{a}_1)^2}{8},
\end{aligned}$$

and

$$\begin{aligned}
&|\Re_1(\zeta_1')(\mathfrak{a}) - \Re_2(\zeta_2')(\mathfrak{a})| \\
&= \Bigg|\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a},\varrho)\mathtt{K}_1(\mathfrak{a},\zeta_1(\varrho),\zeta_1'(\varrho))\,d\varrho \ + \Pi(\mathfrak{a})\right) \\
&\quad - \left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a},\varrho)\mathtt{K}_2(\mathfrak{a},\zeta_2(\varrho),\zeta_2'(\varrho))\ d\varrho \ + \Pi(\mathfrak{a})\right)\Bigg|, \qquad \mathfrak{a} \in [0,a], \\
&= \left|\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{J}(\mathfrak{a},\varrho)(\mathtt{K}_1(\varrho,\zeta(\varrho),\zeta_1'(\varrho)) - \mathtt{K}_2(\mathfrak{a},\zeta_2(\varrho),\zeta_2'(\varrho)))d\varrho\right|, \qquad \mathfrak{a} \in [0,a], \qquad (4.24) \\
&\leq \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |\mathfrak{J}(\mathfrak{a},\varrho)|\,|\mathtt{K}_1(\mathfrak{a},\zeta_1(\varrho),\zeta_{1'}(\varrho)) - \mathtt{K}_2(\mathfrak{a},\zeta_2(\varrho),\zeta_2'(\varrho)))|\,d\varrho, \qquad \mathfrak{a} \in [0,a], \\
&\leq \max_{\varrho\in[0,a]} |\zeta_1(\varrho) - \zeta_2(\varrho)|^2 \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |\mathfrak{J}(\mathfrak{a},\varrho)|d\varrho, \\
&\leq \mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)\frac{1.5(\mathfrak{a}_2-\mathfrak{a}_1)}{2}.
\end{aligned}$$

From (4.23) and (4.24), we easily obtain,

$$\begin{aligned}
\mathsf{d}_{\sigma,\rho}(\Re_1(\zeta_1)(\mathfrak{a}),\Re_2(\zeta_2)(\mathfrak{a})) &\leq \frac{1.5(\mathfrak{a}_2-\mathfrak{a}_1)^2}{8} + \frac{1.5(\mathfrak{a}_2-\mathfrak{a}_1)}{2}\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2) \\
&\leq \frac{9}{10}\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2).
\end{aligned}$$

By applying $\theta(\mathsf{t}) = e^{\sqrt{\mathsf{t}e^{\mathsf{t}}}}$,

$$e^{\sqrt{\mathsf{d}_{\sigma,\rho}(\Re_1(\zeta_1)(\mathfrak{a}),\Re_2(\zeta_2)(\mathfrak{a}))e^{\mathsf{d}_{\sigma,\rho}(\Re_1(\zeta_1)(\mathfrak{a}),\Re_2(\zeta_2)(\mathfrak{a}))}}} \leq e^{\sqrt{\frac{9}{10}\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)e^{\frac{9}{10}\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)}}}$$

$$\leq e^{\sqrt{\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)e^{\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)}}}.$$

Therefore,

$$\mathsf{d}_{\sigma,\rho}(\Re_1(\zeta_1)(\mathfrak{a}),\Re_2(\zeta_2)(\mathfrak{a}))e^{\sqrt{\mathsf{d}_{\sigma,\rho}(\Re_1(\zeta_1)(\mathfrak{a}),\Re_2(\zeta_2)(\mathfrak{a}))e^{\mathsf{d}_{\sigma,\rho}(\Re_1(\zeta_1)(\mathfrak{a}),\Re_2(\zeta_2)(\mathfrak{a}))}}}$$
$$\leq \frac{9}{10}\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)e^{\sqrt{\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)e^{\mathsf{d}_{\sigma,\rho}(\zeta_1,\zeta_2)}}}.$$

So $\exists$ a common FP ζ^* for $\Re_1$ and $\Re_2$. □

4.6 Conclusion

(*i*): Berinde and Pacurar [52] introduced almost contractions which form a class of generalized contractions that includes several contractive type mappings like usual contractions, Kannan mappings, and Zamfirescu mappings. While Azmi [82] introduced two new types of generalized contraction mappings in double controlled MSs.

(*ii*): In this chapter two new types of generalized contraction mappings in double controlled MSs are introduced.

(*iii*): First, we introduced Θ-fuzzy double controlled contraction mapping, which was influenced by the work of [52].

(*iv*): Secondly, we presented fuzzy almost generalized double controlled contraction mapping of the Θ-type, which is an inspiration from the article of Azmi [82].

(*v*): Theorems establishing the existence and uniqueness of the FP for the above-mentioned contractions are presented on complete double controlled MSs.

(*vi*): Some non-trivial examples are provided.

(*vii*): we concluded the chapter by an application for the existence of a solution to the second-order non-linear boundary value problem by using the axioms of the proven results.

Chapter 5

Fuzzy FP Results via Integral Contraction

In this chapter a new class of fuzzy FP theorems for FS-valued mappings, formulated through integral-type Θ-contractions in the context of b-MSs is presented. To enhance the understanding of the theoretical contributions, several illustrative examples are included. In addition, the practical relevance of the established results is highlighted through their application to stochastic Volterra integral equations, thereby reinforcing the validity and utility of the findings.

5.1 Chapter Layout

To facilitate a clearer understanding of the material covered in this chapter, the following structured outline has been adopted:

(i): The chapter begins with a collection of fundamental definitions that form the basis for the concepts and results developed in the following sections.

(ii): Section 5.2 introduces fuzzy θ-type generalized almost contraction mappings in complete b-MSs, and establishes a corresponding FP theorem. Several corollaries are derived, and an illustrative example is provided to support the theoretical findings.

(iii): Section 5.3 discusses various implications of the main FP theorem, presenting several corollaries.

(iv): Section 5.4 applies the obtained theoretical results to establish the existence of solutions for Volterra stochastic integral equations.

(v): Lastly, Section 5.5 presents concluding remarks, summarizing the key contributions and providing a comprehensive overview of the chapter.

Definition 5.1.1. Let κ be the class of functions $\triangle : [0,\infty) \longrightarrow [0,\infty)$ so that:

(i): $\triangle$ is a Lebesgue integrable function and summable on each compact subset of $[0,\infty)$;

(ii): $\int_0^{\mathrm{t}} \triangle(\varsigma)\mathrm{d}(\varsigma) > 0$ for each $\mathrm{t} > 0$. [104]

Definition 5.1.2. Let $\tilde{\rho}$ be the set of strictly increasing functions in a b-MS, $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$(\tilde{\rho}_1)$: $\sum_{n=0}^{\infty} s^n\varphi^n < +\infty$, where φ^n denotes the nth iterate of φ;

$(\tilde{\rho}_2)$: $\varphi(\mathrm{t}) < \mathrm{t}\ \forall\ \mathrm{t} > 0$ and $\varphi(0) = 0$.

Then, it is called a comparison function on b-MS. [105]

Lemma 5.1.3. Assume $\eth$ be a b-metric linear space with $s \geq 0$. If $\Re_1, \Re_2 : \eth \longrightarrow \mathfrak{D}(\eth)$ be a FM and $\mathrm{v}_0 \in \eth$ then $\exists\ \mathrm{v}_1 \in \mathcal{V}$ so that $\{\mathrm{v}_1\} \subset \Re_1(\mathrm{v}_0)$

$$\eth(\mathrm{v}, \Re_2) \leq s\mathrm{H}(\Re_1, \Re_2).$$

Lemma 5.1.4. If $\{\mathsf{P}_n\}$ is a sequence in $[0,\infty)$ and $\psi \in E$ then

$$\lim_{n\longrightarrow\infty} \int_0^{\mathrm{P}_n} \psi(\varsigma)\mathrm{d}(\varsigma) = 0 \;\Leftrightarrow\; \mathrm{P}_n \longrightarrow 0, \text{ as } n \longrightarrow \infty. \text{ [106]}$$

Lemma 5.1.5. Assume $(\mathsf{G}, \mathsf{d}_b)$ being a complete b-MS with $s \geq 0$. For $\mathsf{A}, \mathsf{B} \in \mathfrak{CB}(\mathsf{G})$ and $\mathrm{v} \in \mathsf{A}$

$$\mathsf{d}_b(\mathrm{v}, \mathsf{B}) \leq s\mathrm{H}(\mathsf{A}, \mathsf{B}). \text{ [23]}$$

5.2 Fuzzy Θ-Type Generalized Almost Contraction

Throughout the chapter d_b is considered as complete b-MS. Motivated by the idea of Kanwal et al. [81] we will introduced fuzzy Θ-type generalized almost contraction mappings(F_ΘGAC) in b-MS as follows:

Definition 5.2.1. Suppose that $(\mathsf{G}, \mathsf{d}_b)$ be a b-MS with $s \geq 1$, then the pair $\Re_1, \Re_2 : \mathsf{G} \to \mathcal{F}(\mathsf{G})$ is said to be F_ΘGAC if $\forall\, \zeta_1, \zeta_2 \in \mathsf{G}$, $[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}$ are non empty elements of $\mathfrak{CB}(\mathsf{G})$ with $\alpha_{\Re_1(\zeta_1)}, \alpha_{\Re_2(\zeta_2)} \in (0, 1]$, such that
$\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) > 0$

$$\Rightarrow \int_0^{[\theta(s\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\theta(\mathtt{M}(\zeta_1,\zeta_2))]^k + \mathfrak{L}\Xi(\zeta_1,\zeta_2)} \triangle(\varsigma)\mathrm{d}(\varsigma), \tag{5.1}$$

with

$$\begin{aligned}
&\mathtt{M}(\zeta_1, \zeta_2) \\
&= \max\Big\{\mathsf{d}_b(\zeta_1, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}), \mathsf{d}_b(\zeta_1, \zeta_2), \mathsf{d}_b(\zeta_2, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}), \\
&\frac{\{\mathsf{d}_b(\zeta_1, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) + \mathsf{d}_b(\zeta_2, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}})\}}{2s}\Big\},
\end{aligned}$$

and

$$\begin{aligned}
&\Xi(\zeta_1, \zeta_2) \\
&= \min\Big\{\mathsf{d}_b(\zeta_1, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}), \mathsf{d}_b(\zeta_2, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}), \\
&\mathsf{d}_b(\zeta_1, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}), \mathsf{d}_b(\zeta_2, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}})\Big\},
\end{aligned}$$

and $\theta \in \Theta_s, \mathfrak{L} \geq 0$.

Theorem 5.2.2. Let $(\mathsf{G}, \mathsf{d}_b)$ be a complete b-MS with $s \geq 1$ and $\Re_1, \Re_2$ be the pair of F_ΘGAC. Then, $\exists\, \zeta^* \in \mathsf{G}$ such that $\zeta^* \in [\Re_1(\zeta^*)]_{\alpha_{\Re_1(\zeta^*)}} \cap [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}$.

Proof. Let $\zeta_0 \in \mathsf{G}$, by hypothesis $\exists\, \alpha_{\Re_1(\zeta_0)} \in (0, 1]$ such that $[\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}$ is non empty and $\in \mathfrak{CB}(\mathsf{G})$. Let $\zeta_1 \in [\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}$ and $\exists\, \alpha_{\Re_2(\zeta_1)} \in (0, 1]$ such that

$[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}$ is non empty and $\in \mathfrak{CB}(\mathsf{G})$. By using Lemma 5.1.5 and Definition 5.2.1.

$$[\theta(\mathsf{d}_b(\zeta_1, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}))] \leq [\theta(\mathrm{H}([\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}}, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}))]$$

$$\Rightarrow \int_0^{[\theta(s\mathsf{d}_b(\zeta_1,[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\theta(s\mathrm{H}([\Re_1(\zeta_0)]_{\alpha_{\Re_1(\zeta_0)}},[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}))]} \triangle(\varsigma)\mathrm{d}(\varsigma)$$
$$\leq \int_0^{[\theta(\mathtt{M}(\zeta_0,\zeta_1))]^k+\mathfrak{L}\Xi(\zeta_0,\zeta_1)} \triangle(\varsigma)\mathrm{d}(\varsigma). \tag{5.2}$$

Since, by definition

$$\theta(\mathsf{d}_b(\zeta_1, [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}})) = \inf_{\mathrm{f}_1\in[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}} \theta(\mathsf{d}_b(\zeta_1, \mathrm{f}_1)).$$

Thus, by Θ_s-contraction

$$\inf_{\mathrm{f}_1\in[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}} [\theta(\mathsf{d}_b(\zeta_1, \mathrm{f}_1))] \leq [\theta(\mathtt{M}(\zeta_0, \zeta_1))]^k + \mathfrak{L}\Xi(\zeta_0, \zeta_1).$$

Now $\exists\ \zeta_2 \in [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}$. So (5.2) becomes,

$$\int_0^{[\theta(\mathsf{d}_b(\zeta_1,\zeta_2))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\theta(\mathtt{M}(\zeta_0,\zeta_1))]^k+\mathfrak{L}\Xi(\zeta_0,\zeta_1)} \triangle(\varsigma)\mathrm{d}(\varsigma). \tag{5.3}$$

Now,

$$\mathtt{M}(\zeta_0, \zeta_1) \leq \max\Big\{\mathsf{d}_b(\zeta_0, \zeta_1), \mathsf{d}_b(\zeta_0, \zeta_1), \mathsf{d}_b(\zeta_1, \zeta_2),$$

$$\frac{\{\mathsf{d}_b(\zeta_0, \zeta_2) + \mathsf{d}_b(\zeta_1, \zeta_1)\}}{2s}\Big\}.$$

$$\frac{\{\mathsf{d}_b(\zeta_0, \zeta_2) + \mathsf{d}_b(\zeta_1, \zeta_1)\}}{2s}\Big\}.$$

$$\Rightarrow \mathtt{M}(\zeta_0, \zeta_1) \leq \max\Big\{\mathsf{d}_b(\zeta_0, \zeta_1), \mathsf{d}_b(\zeta_1, \zeta_2)\Big\}.$$

and

$$\Xi(\zeta_0,\zeta_1) = \min\Big\{\mathsf{d}_b(\zeta_0,\zeta_1),\mathsf{d}_b(\zeta_1,\zeta_2),\mathsf{d}_b(\zeta_0,\zeta_2),\mathsf{d}_b(\zeta_1,\zeta_1)\Big\},$$
$$=\mathsf{d}_b(\zeta_1,\zeta_1)$$
$$=0.$$

If we take,

$$\max\Big\{\mathsf{d}_b(\zeta_0,\zeta_1),\mathsf{d}_b(\zeta_1,\zeta_2)\Big\} = \mathsf{d}_b(\zeta_1,\zeta_2).$$

Then (5.3) becomes,

$$\int_0^{[\theta(s\mathsf{d}_b(\zeta_1,\zeta_2))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\theta(\mathsf{d}_b(\zeta_1,\zeta_2))]^k} \triangle(\varsigma)\mathrm{d}(\varsigma).$$

a contradiction.

Thus,

$$\max\Big\{\mathsf{d}_b(\zeta_0,\zeta_1),\mathsf{d}_b(\zeta_1,\zeta_2)\Big\} = \mathsf{d}_b(\zeta_0,\zeta_1).$$

So (5.3) becomes,

$$\int_0^{[\theta(s\mathsf{d}_b(\zeta_1,\zeta_2))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\theta(\mathsf{d}_b(\zeta_0,\zeta_1))]^k} \triangle(\varsigma)\mathrm{d}(\varsigma).$$

Now $\exists\ \alpha_{\Re_1(\zeta_2)} \in (0,1]$ such that $[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}$ is non empty and $\in \mathfrak{CB}(\mathsf{G})$. By Lemma 5.1.5 and and Definition 5.2.1, we have

$$\int_0^{[\theta(s\mathsf{d}_b([\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}},\zeta_2))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\theta(s\mathrm{H}([\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}},[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}))]} \triangle(\varsigma)\mathrm{d}(\varsigma)$$
$$\leq \int_0^{[\theta(\mathtt{M}(\zeta_1,\zeta_2))]^k+\mathfrak{L}\Xi(\zeta_1,\zeta_2)} \triangle(\varsigma)\mathrm{d}(\varsigma). \tag{5.4}$$

Since, by definition of Θ_s-contraction

$$[\theta(\mathsf{d}_b(\zeta_2, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}))] = \inf_{\mathrm{f}_2 \in [\Re_1(\zeta_2)]_{\alpha_{\Re_1}(\zeta_2)}} \theta(\mathsf{d}_b(\zeta_2, \mathrm{f}_2)).$$

Thus, by Definition 5.2.1

$$\inf_{\mathrm{f}_2 \in [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}} \theta(\mathsf{d}_b(\zeta_2, \mathrm{f}_2)) \leq [\theta(\mathsf{d}_b(\zeta_1, \zeta_2))]^{\mathrm{k}} + \mathfrak{L}\Xi(\zeta_1, \zeta_2).$$

Now $\exists\ \zeta_3 \in [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}$. So (5.4) becomes,

$$\int_0^{[\theta(s\mathsf{d}_b(\zeta_2,\zeta_3))]} \triangle\,(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\theta(\mathtt{M}(\zeta_1,\zeta_2))]^k + \mathfrak{L}\Xi(\zeta_1,\zeta_2)} \triangle\,(\varsigma)\mathrm{d}(\varsigma). \tag{5.5}$$

Now

$$\begin{aligned}\mathtt{M}(\zeta_1, \zeta_2) \leq & \max\Big\{\mathsf{d}_b(\zeta_1, \zeta_2), \mathsf{d}_b(\zeta_2, \zeta_3), \mathsf{d}_b(\zeta_1, \zeta_2), \\ & \frac{\{\mathsf{d}_b(\zeta_1, \zeta_3) + \mathsf{d}_b(\zeta_2, \zeta_2)\}}{2s}\Big\}.\end{aligned}$$

$$\Rightarrow \mathtt{M}(\zeta_1, \zeta_2) = \max\Big\{\mathsf{d}_b(\zeta_1, \zeta_2), \mathsf{d}_b(\zeta_2, \zeta_3)\Big\}.$$

and

$$\begin{aligned}\Xi(\zeta_1, \zeta_2) =& \min\Big\{\mathsf{d}_b(\zeta_2, \zeta_3), \mathsf{d}_b(\zeta_1, \zeta_2), \mathsf{d}_b(\zeta_2, \zeta_2), \mathsf{d}_b(\zeta_1, \zeta_3)\Big\}, \\ \Xi(\zeta_1, \zeta_2) =& \mathsf{d}_b(\zeta_2, \zeta_2) \\ =& 0.\end{aligned}$$

If we take,

$$\max\Big\{\mathsf{d}_b(\zeta_1, \zeta_2), \mathsf{d}_b(\zeta_2, \zeta_3)\Big\} = \mathsf{d}_b(\zeta_2, \zeta_3).$$

Then, (5.5) becomes,

$$\int_0^{[\theta(s\mathsf{d}_b(\zeta_2,\zeta_3))]} \triangle\,(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\theta(\mathsf{d}_b(\zeta_2,\zeta_3))]^k} \triangle\,(\varsigma)\mathrm{d}(\varsigma)$$

a contradiction. Thus,

$$\max\left\{\mathsf{d}_b(\zeta_1,\zeta_2),\mathsf{d}_b(\zeta_2,\zeta_3)\right\}=\mathsf{d}_b(\zeta_1,\zeta_2).$$

So (5.5) implies,

$$\int_0^{[\theta(s\mathsf{d}_b(\zeta_2,\zeta_3))]}\triangle(\varsigma)\mathrm{d}(\varsigma)\leq\int_0^{[\theta(\mathsf{d}_b(\zeta_1,\zeta_2))]^k}\triangle(\varsigma)\mathrm{d}(\varsigma).$$

Using this procedure, we generate a sequence $\{\zeta_n\}\in\mathsf{G}\ \ \forall\ n\in\mathbb{N}$ such that $\zeta_{2n+2}\in[\Re_2(\zeta_{2n+1})]_{\alpha_{\Re_2(\zeta_{2n+1})}}$ and $\zeta_{2n+1}\in[\Re_1(\zeta_{2n})]_{\alpha_{\Re_1(\zeta_{2n})}}$ such that

$$\int_0^{[\theta(s\mathsf{d}_b(\zeta_{2n+1},\zeta_{2n+2}))]}\triangle(\varsigma)\mathrm{d}(\varsigma)\leq\int_0^{[\theta(\mathsf{d}_b(\zeta_{2n},\zeta_{2n+1}))]^k}\triangle(\varsigma)\mathrm{d}(\varsigma).\tag{5.6}$$

Also,

$$\int_0^{[\theta(s\mathsf{d}_b(\zeta_{2n+2},\zeta_{2n+3}))]}\triangle(\varsigma)\mathrm{d}(\varsigma)\leq\int_0^{[\theta(\mathsf{d}_b(\zeta_{2n+1},\zeta_{2n+2}))]^k}\triangle(\varsigma)\mathrm{d}(\varsigma).\tag{5.7}$$

From (5.6) and (5.7) and by using Θ_{4s} we obtain

$$\int_0^{[\theta(s^n\mathsf{d}_b(\zeta_{n+1},\zeta_{n+1}))]}\triangle(\varsigma)\mathrm{d}(\varsigma)\leq\int_0^{[\theta(s^{n-1}\mathsf{d}_b(\zeta_{n-1},\zeta_n))]^k}\triangle(\varsigma)\mathrm{d}(\varsigma).\tag{5.8}$$

Therefore,

$$\begin{aligned}\int_0^{[\theta(s^n\mathsf{d}_b(\zeta_n,\zeta_{n+1}))]}\triangle(\varsigma)\mathrm{d}(\varsigma)&\leq\int_0^{[\theta(s^{n-1}\mathsf{d}_b(\zeta_{n-1},\zeta_n))]^k}\triangle(\varsigma)\mathrm{d}(\varsigma)\\&\leq\int_0^{[\theta(s^{n-2}\mathsf{d}_b(\zeta_{n-2},\zeta_{n-1}))]^{k^2}}\triangle(\varsigma)\mathrm{d}(\varsigma)\\&\leq\int_0^{[\theta(s^{n-3}\mathsf{d}_b(\zeta_{n-3},\zeta_{n-2}))]^{k^3}}\triangle(\varsigma)\mathrm{d}(\varsigma)\\&\ \ \vdots\\&\leq\int_0^{[\theta(\mathsf{d}_b(\zeta_0,\zeta_1))]^{k^n}}\triangle(\varsigma)\mathrm{d}(\varsigma).\end{aligned}$$

Since, $\theta\in\Theta_s$, hence

$$\lim_{n\longrightarrow\infty}[\theta(s^n\mathsf{d}_b(\zeta_n,\zeta_{n+1}))]=1.$$

Thus, by Θ_{2s}

$$\lim_{n\longrightarrow\infty} s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1}) = 0^+.$$

In the view Θ_{3s}, $\exists$ $\mathsf{p} \in (0,1)$ and $\mathsf{r} \in (0,\infty]$ so that

$$\lim_{n\longrightarrow\infty} \frac{\theta(s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1})) - 1}{(s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1}))^{\mathsf{p}}} = \mathsf{r}.$$

Case 1.

Let $\mathsf{r} < \infty$ and $\frac{1}{2} = \mathcal{C}_1 > 0$. Hence $\exists$ $n_0 \in \mathbb{N}$ such that $\forall$ $n > n_0$

$$\left| \frac{\theta(s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1})) - 1}{(s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1})^{\mathsf{p}}} - \mathsf{r} \right| \leq \mathcal{C}_1.$$

That is

$$\frac{\theta(s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1})) - 1}{(s^n \varphi^n \mathsf{d}_b(\zeta_n, \zeta_{n+1}))^{\mathsf{p}}} - \mathsf{r} \geq \mathsf{r} - \mathcal{C}_1 = \mathcal{C}_1.$$

Then,

$$n[s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1})]^{\mathsf{p}} \leq \frac{n}{\mathcal{C}_1}[\theta(s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1}))) - 1].$$

Case 2.

Let $\mathsf{r} = \infty$ and $\mathcal{C}_1 > 0$. Then $\exists$ $n_0 \in \mathbb{N}$ such that $\forall$ $n > n_0$

$$\mathcal{C}_1 \leq \frac{\theta(s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1})) - 1}{(s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1}))^{\mathsf{p}}},$$

which gives us

$$n[s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1})]^{\mathsf{p}} \leq \frac{n}{\mathcal{C}_1}[\theta(s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1})) - 1].$$

In both cases $\frac{1}{\mathcal{C}_1} > 0$, and $n_0 \in \mathbb{N}$ so that $\forall$ $n > n_0$,

$$n[s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1})]^{\mathsf{p}} \leq \frac{n}{\mathcal{C}_1}([\theta(\mathsf{d}_b(\zeta_n, \zeta_{n+1}))] - 1).$$

Now, we have

$$n[s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1})]^{\mathsf{p}} \leq \frac{n}{\mathcal{C}_1}([\theta(\mathsf{d}_b(\zeta_0, \zeta_1))]^{k^n} - 1).$$

Since, $\theta \in \Theta_s$, hence

$$\lim_{n \longrightarrow \infty} n[s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1})]^{\mathsf{p}} = 0.$$

Since $\exists\ n_1 \ni \forall\ n > n_1$,

$$n[s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1})]^{\mathsf{p}} \leq 1.$$

This implies that

$$s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1}) \leq \frac{1}{n^{\frac{1}{\mathsf{p}}}}$$

$$\int_0^{[s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1})]} \triangle(\varsigma) \mathrm{d}(\varsigma) \leq \int_0^{\frac{1}{n^{\frac{1}{\mathsf{p}}}}} \triangle(\varsigma) \mathrm{d}(\varsigma).$$

Now to prove $\{\zeta_n\}$ is a Cauchy sequence, let $m, n \in \mathbb{N}$ such that $m > n > n_0$, by using triangular inequality

$$\mathsf{d}_b(\zeta_n, \zeta_m) \leq s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1}) + s^{n+1} \mathsf{d}_b(\zeta_{n+1}, \zeta_{n+2}) + s^{n+2} \mathsf{d}_b(\zeta_{n+2}, \zeta_{n+3}) +$$

$$\cdots + s^{m-1} \mathsf{d}_b(\zeta_{m-1}, \zeta_m).$$

From above, we can write as

$$\int_0^{\mathsf{d}_b(\zeta_n, \zeta_m)} \triangle(\varsigma) \mathrm{d}(\varsigma) \leq \Big[\int_0^{s^n \mathsf{d}_b(\zeta_n, \zeta_{n+1})} \triangle(\varsigma) \mathrm{d}(\varsigma) + \int_0^{s^{n+1} \mathsf{d}_b(\zeta_{n+1}, \zeta_{n+2})} \triangle(\varsigma) \mathrm{d}(\varsigma) +$$

$$\int_0^{sn+2 \mathsf{d}_b(\zeta_{n+2}, \zeta_{n+3})} \triangle(\varsigma) \mathrm{d}(\varsigma) + \cdots + \int_0^{s^{m-1} \mathsf{d}_b(\zeta_{m-1}, \zeta_m)} \triangle(\varsigma) \mathrm{d}(\varsigma)\Big]$$

$$\begin{aligned} \int_0^{\mathsf{d}_b(\zeta_n, \zeta_m)} \triangle(\varsigma) \mathrm{d}(\varsigma) &\leq \sum_{i=n}^{m-1} \int_0^{s^i \varphi^i \mathsf{d}_b(\zeta_n, \zeta_{n+1})} \triangle(\varsigma) \mathrm{d}(\varsigma) \\ &\leq \sum_{i=n}^{m-1} \int_0^{\frac{1}{n^{\frac{1}{\mathsf{p}}}}} \triangle(\varsigma) \mathrm{d}(\varsigma) \\ &\leq \sum_{i=n}^{\infty} \int_0^{\frac{1}{n^{\frac{1}{\mathsf{p}}}}} \triangle(\varsigma) \mathrm{d}(\varsigma). \end{aligned}$$

Since $0 < \mathsf{p} < 1$, the series $\sum_{i=n}^{\infty} \int_0^{\frac{1}{n^{\frac{1}{\mathsf{p}}}}} \triangle(\varsigma)\mathrm{d}(\varsigma)$ converges when $m, n \longrightarrow \infty$, we get $\mathsf{d}_b(\zeta_n, \zeta_m) \longrightarrow 0$. So it is clear that $\{\zeta_n\} \in \mathsf{G}$ is a Cauchy sequence in $(\mathsf{G}, \mathsf{d}_b)$. By completeness $\exists\, \zeta^* \ni \{\zeta_n\} \longrightarrow \zeta^*$. Next we claim that $\{\zeta^*\} \in [\Re_2(\zeta^*)]_{\alpha_{\Re_2}(\zeta^*)}$. Assume on contrary that ζ^* does not belong to $[\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}$ (that is, $\mathsf{d}_b(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}) > 0$), then there are $n_0 \in \mathbb{N}$ and a subsequence $\{\zeta_{n_k}\}$ of ζ_n so that

$$\mathsf{d}_b(\zeta_{2n_k+1}, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}) > 0, \ \forall\ n_k \geq n_0.$$

Since $\mathsf{d}_b(\zeta_{2n_k+1}, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}) > 0$, we have

$$[\theta(\mathsf{d}_b(\zeta_{2n_k+1}, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}))] \leq [\theta(\mathrm{H}([\Re_1(\zeta_{2n_k})]_{\alpha_{\Re_1(\zeta_{2n_k})}}, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}))]. \quad (5.9)$$

$$\begin{aligned} &\Rightarrow \int_0^{[\theta(s\mathsf{d}_b(\zeta_{2n_k+1}, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}))]} \triangle\,(\varsigma)\mathrm{d}(\varsigma) \\ &\leq \int_0^{[\theta(s\mathrm{H}([\Re_1(\zeta_{2n_k})]_{\alpha_{\Re_1(\zeta_{2n_k})}}, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}))]} \triangle\,(\varsigma)\mathrm{d}(\varsigma) \qquad (5.10) \\ &\leq \int_0^{[\theta(s\mathsf{d}_b(\zeta_{2n_k+1}, \zeta^*))]^k} \triangle\,(\varsigma)\mathrm{d}(\varsigma). \end{aligned}$$

Now

$$\begin{aligned} &\mathtt{M}(\zeta_{2n_k}, \zeta^*) \\ &= \max\Big\{\mathsf{d}_b(\zeta_{2n_k}, \zeta^*), \mathsf{d}_b(\zeta_{2n_k}, [\Re_1(\zeta_{2n_k})]_{\alpha_{\Re_1(\zeta_{2n_k})}}), \mathsf{d}_b(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}), \\ &\quad \frac{\{\mathsf{d}_b(\zeta_{2n_k}, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}) + \mathsf{d}_b(\zeta^*, [\Re_1(\zeta_{2n_k})]_{\alpha_{\Re_1(\zeta_{2n_k})}})\}}{2s}\Big\}. \\ &\leq \max\Big\{\mathsf{d}_b(\zeta_{2n_k}, \zeta^*), \mathsf{d}_b(\zeta_{2n_k}, \zeta_{2n_k+1}), \mathsf{d}_b(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}), \\ &\quad \frac{\{\mathsf{d}_b(\zeta_{2n_k}, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}) + \mathsf{d}_b(\zeta^*, \zeta_{2n_k+1})\}}{2s}\Big\}. \end{aligned}$$

$$\begin{aligned} &\Xi(\zeta_{2n_k}, \zeta^*) \\ &= \min\Big\{\mathsf{d}_b(\zeta_{2n_k}, [\Re_1(\zeta_{2n_k})]_{\alpha_{\Re_1(\zeta_{2n_k})}}), \mathsf{d}_b(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}), \\ &\quad \mathsf{d}_b(\zeta_{2n_k}, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}) + \mathsf{d}_b(\zeta^*, [\Re_1(\zeta_{2n_k})]_{\alpha_{\Re_1(\zeta_{2n_k})}})\Big\}. \end{aligned}$$

$$\begin{aligned} \Xi(\zeta_{2n_k}, \zeta^*) &\leq \min\Big\{\mathsf{d}_b(\zeta_{2n_k}, \zeta_{2n_k+1}), \mathsf{d}_b(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}), \\ &\quad \mathsf{d}_b(\zeta_{2n_k}, [\Re_2(\zeta^*)]_{\alpha_{\Re_2}(\zeta^*)}) + \mathsf{d}_b(\zeta^*, \zeta_{2n_k+1})\Big\}. \end{aligned}$$

Letting n $\longrightarrow \infty$ and using the continuity of θ, equation (5.10) becomes,

$$\int_0^{[\theta(s\mathsf{d}_b(\zeta^*,[\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\theta(\mathsf{d}_b(\zeta^*,[\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}))]^k} \triangle(\varsigma)\mathrm{d}(\varsigma),$$

this leads to a contradiction. Hence

$$\mathsf{d}_b(\zeta^*, [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}) = 0,$$

and $\zeta^* \in [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}}$. Using a similar approach, it can be shown that $\zeta^* \in [\Re_1(\zeta^*)]_{\alpha_{\Re_1(\zeta^*)}}$.

Therefore $\zeta^* \in [\Re_2(\zeta^*)]_{\alpha_{\Re_2(\zeta^*)}} \cap [\Re_1(\zeta^*)]_{\alpha_{\Re_1(\zeta^*)}}$. □

Example 5.2.3. Let $\mathsf{G} = [0,1]$, defined $\mathsf{d}_b : \mathsf{G} \times \mathsf{G} \longrightarrow \mathbb{R}$ as

$$\mathsf{d}_b(\zeta_1, \zeta_2) = |\zeta_1 - \zeta_2|^2,$$

then $(\mathsf{G}, \mathsf{d}_b)$ is a complete b-MS with $s = 2$.

Define $\Re_1, \Re_2 : \mathsf{G} \to \mathcal{F}(\mathsf{G})$ by,

$$\Re_1(\zeta_1)\mathsf{t} = \begin{cases} \alpha, & \text{if } 0 \leq \mathsf{t} \leq \frac{\zeta_1}{15}, \\ \frac{\alpha}{15}, & \text{if } \frac{\zeta_1}{15} \leq \mathsf{t} \leq \frac{\zeta_1}{10}, \\ \frac{\alpha}{10}, & \text{if } \frac{\zeta_1}{10} \leq \mathsf{t} \leq \frac{\zeta_1}{5}, \\ \frac{\alpha}{5}, & \text{if } \frac{\zeta_1}{5} \leq \mathsf{t} \leq 1. \end{cases}$$

$$\Re_2(\zeta_2)\mathsf{t} = \begin{cases} \alpha, & \text{if } 0 \leq \mathsf{t} \leq \frac{\zeta_2}{30}, \\ \frac{\alpha}{30}, & \text{if } \frac{\zeta_2}{30} \leq \mathsf{t} \leq \frac{\zeta_2}{20}, \\ \frac{\alpha}{20}, & \text{if } \frac{\zeta_2}{20} \leq \mathsf{t} \leq \frac{\zeta_2}{10}, \\ \frac{\alpha}{10}, & \text{if } \frac{\zeta_2}{10} \leq \mathsf{t} \leq 1. \end{cases}$$

Now for $\alpha_{\Re_2}$ and $\alpha_{\Re_1} = 1$ we have

$$[\Re_1(\zeta_1)]_\alpha = [0, \frac{\zeta_1}{15}],$$
$$[\Re_2(\zeta_2)]_\alpha = [0, \frac{\zeta_2}{30}].$$

Since

$$\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) = |\tfrac{\zeta_1}{15} - \tfrac{\zeta_2}{30}|^2 > 0 \text{ for } \zeta_1 \neq \zeta_2.$$

Now

$$\begin{aligned}
&\mathtt{M}(\zeta_1, \zeta_2)\\
&= \max\Bigg\{ \mathsf{d}_b(\zeta_1, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}), \mathsf{d}_b(\zeta_1, \zeta_2), \mathsf{d}_b(\zeta_2, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}),\\
&\frac{\{\mathsf{d}_b(\zeta_1, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) + \mathsf{d}_b(\zeta_2, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}})\}}{2s} \Bigg\}.
\end{aligned}$$

$$\begin{aligned}
\mathtt{M}(\zeta_1, \zeta_2) = \max\Bigg\{&|\zeta_1 - \zeta_2|^2, |\zeta_1 - \frac{\zeta_1}{15}|^2, |\zeta_2 - \frac{\zeta_2}{30}|^2,\\
&\frac{\{|\zeta_1 - \frac{\zeta_2}{30}|^2 + |\zeta_2 - \frac{\zeta_1}{15}|^2\}}{5}\Bigg\},\\
&\leq |\zeta_1 - \zeta_2|^2,
\end{aligned}$$

and

$$\begin{aligned}
\Xi(\zeta_1, \zeta_2) = \min\Big\{&\mathsf{d}_b(\zeta_1, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}), \mathsf{d}_b(\zeta_2, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}),\\
&\mathsf{d}_b(\zeta_1, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}), \mathsf{d}_b(\zeta_2, [\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}\Big\}.
\end{aligned}$$

$$\begin{aligned}
\Xi(\zeta_1, \zeta_2) = \min\Big\{&|\zeta_1 - \frac{\zeta_1}{15}|^2, |\zeta_2 - \frac{\zeta_2}{30}|^2,\\
&|\zeta_1 - \frac{\zeta_2}{30}|^2, |\zeta_2 - \frac{\zeta_1}{15}|^2)\Big\}\\
&\leq |\frac{14}{15}\zeta_1|^2.
\end{aligned}$$

Taking $\theta(\mathrm{t}) = 2^{\sqrt[k]{\mathrm{t}}}$, for some $k = \frac{1}{\sqrt{3}} \in (0, 1)$, we have

$$\begin{aligned}
&\int_0^{[\theta(s\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1}(\zeta_1)}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2}(\zeta_2)}))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\theta(\mathtt{M}(\zeta_1,\zeta_2))]^k + \mathfrak{L}\Xi(\zeta_1,\zeta_2)} \triangle(\varsigma)\mathrm{d}(\varsigma), \qquad (5.11)\\
&\Rightarrow \int_0^{[\theta(s(|\frac{\zeta_1}{15} - \frac{\zeta_2}{30}|^2))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\theta(|\zeta_1 - \zeta_2|^2)]^k + |\frac{14}{15}\zeta_1|^2} \triangle(\varsigma)\mathrm{d}(\varsigma),\\
&\leq \int_0^{[\theta(|\zeta_1 - \zeta_2|^2)]^k} \triangle(\varsigma)\mathrm{d}(\varsigma).
\end{aligned}$$

All assumption of Theorem 5.2.2 with $\mathfrak{L} = 1$ are satisfied, and 0 is the FP of $\Re_1$ and $\Re_2$.

If we consider $\mathfrak{L} = 0$ in Definition 5.2.1 then we have the following:

Theorem 5.2.4. Let $(\mathsf{G}, \mathsf{d}_b)$ be a complete b-MS, and let $\Re_1, \Re_2$ be the pair of FMs such that for each $\zeta_1, \zeta_2 \in \mathsf{G}$, $\exists\ \alpha_{\Re_1(\zeta_1)}, \alpha_{\Re_2(\zeta_2)} \in (0,1]$ such that
$[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}} \in \mathfrak{CB}(\mathsf{G})$ are non empty, and $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$,
$\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) > 0$,

$$\Rightarrow \int_0^{[\theta(s\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\theta(\mathtt{M}(\zeta_1,\zeta_2))]^k} \triangle(\varsigma)\mathrm{d}(\varsigma), \quad (5.12)$$

where $\theta \in \Theta_s$ and $\mathtt{M}(\zeta_1, \zeta_2)$ is same as in Definition 5.2.1.
Then, $\exists$ a common FP of $\Re_1$ and $\Re_2$.

If we take $\triangle(\varsigma) = 1$ in Definition 5.2.1 we get the following:

Corollary 5.2.5. Let $(\mathsf{G}, \mathsf{d}_b)$ be a complete b-MS, and let $\Re_1, \Re_2$ be the pair of FMs such that for each $\zeta_1, \zeta_2 \in \mathsf{G}$, $\exists\ \alpha_{\Re_1(\zeta_1)}, \alpha_{\Re_2(\zeta_2)} \in (0,1]$ such that
$[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}} \in \mathfrak{CB}(\mathsf{G})$ are non empty, and $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$,
$\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) > 0$,

$$\Rightarrow [\theta(s\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}))] \leq [\theta(\mathtt{M}(\zeta_1, \zeta_2))]^k + \mathfrak{L}\Xi(\zeta_1, \zeta_2),$$

where $\theta \in \Theta_s$, $\mathfrak{L} \geq 0$ and $\mathtt{M}(\zeta_1, \zeta_2)$ is same as in Definition 5.2.1.
Then, $\exists\ \mathrm{f} \in \mathsf{G}$ such that $\mathrm{f} \in [\Re_1(\mathrm{f})]_{\alpha_{\Re_1(\mathrm{f})}} \cap [\Re_2(\mathrm{f})]_{\alpha_{\Re_2(\mathrm{f})}}$.

By taking $\Re_2 = \Re_1$ in Definition 5.2.1 we have the following:

Corollary 5.2.6. Assume $(\mathsf{G}, \mathsf{d}_b)$ be a complete b-MS and $\Re_1 : \mathsf{G} \to \mathcal{F}(\mathsf{G})$ be a FM such that for each $\zeta_1 \in \mathsf{G}$, $\exists\ \alpha_{\Re_1(\zeta_1)}, \alpha_{\Re_2(\zeta_2)} \in (0,1]$ such that $[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}$ is non empty and $\in \mathfrak{CB}(\mathsf{G})$, and $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$,
$\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}) > 0$, implies

$$\int_0^{[\theta(s\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\theta(\mathtt{M}(\zeta_1,\zeta_2))]^k + \mathfrak{L}\Xi(\zeta_1,\zeta_2)} \triangle(\varsigma)\mathrm{d}(\varsigma), \quad (5.13)$$

where

$$\begin{aligned}
&\mathtt{M}(\zeta_1,\zeta_2)\\
&=\max\Bigg\{\mathsf{d}_b(\zeta_1,[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}),\mathsf{d}_b(\zeta_2,[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}),\mathsf{d}_b(\zeta_1,\zeta_2),\\
&\frac{\{\mathsf{d}_b(\zeta_1,[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}})+\mathsf{d}_b(\zeta_2,[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}})\}}{2s}\Bigg\},
\end{aligned}$$

and

$$\begin{aligned}
\Xi(\zeta_1,\zeta_2)=\min\Big\{&\mathsf{d}_b(\zeta_1,[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}),\mathsf{d}_b(\zeta_2,[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}),\\
&\mathsf{d}_b(\zeta_1,[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}),\mathsf{d}_b(\zeta_2,[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}})\Big\},\quad \theta\in\Theta_s,\mathfrak{L}\geq 0.
\end{aligned}$$

Then $\Re_1$ has a FP in G.

If we take $\mathfrak{L}=0$ in Corollary 5.2.6 we get the following:

Corollary 5.2.7. Assume $(\mathsf{G},\mathsf{d}_b)$ be a complete b-MS and $\Re_1:\mathsf{G}\to\mathcal{F}(\mathsf{G})$ be a FM if for each $\zeta_1\in\mathsf{G}\ \exists\ \alpha_{\Re_1(\zeta_1)},\alpha_{\Re_2(\zeta_2)}\in(0,1]$ such that $[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}$ is non empty and $\in\mathfrak{CB}(\mathsf{G})$ and $\forall\ \zeta_1,\zeta_2\in\mathsf{G}$,
$\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}},[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}})>0$, implies

$$\int_0^{[\theta(s\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}},[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}}))]}\triangle(\varsigma)\mathrm{d}(\varsigma)\leq\int_0^{[\theta(\mathtt{M}(\zeta_1,\zeta_2))]^k}\triangle(\varsigma)\mathrm{d}(\varsigma),\qquad(5.14)$$

where $\theta\in\Theta_s$ and $\mathtt{M}(\zeta_1,\zeta_2)$ is same as in Corollary 5.2.6.
Then $\exists$ f $\in\mathsf{G}$ such that f $\in[\Re_1(\mathrm{f})]_{\alpha_{\Re_1(\mathrm{f})}}$.

If we take $\triangle(\varsigma)=1$ in Corollary 5.2.7 we get the following:

Corollary 5.2.8. Assume $(\mathsf{G},\mathsf{d}_b)$ be a complete b-MS and $\Re_1:\mathsf{G}\to\mathcal{F}(\mathsf{G})$ be a FM if for each $\zeta_1\in\mathsf{G}\ \exists\ \alpha_{\Re_1(\zeta_1)},\alpha_{\Re_2(\zeta_2)}\in(0,1]$ such that $[\Re_1(\zeta_1)]_{\alpha_{\Re_1}(\zeta_1)}$ is non empty and $\in\mathfrak{CB}(\mathsf{G})$ and $\forall\ \zeta_1,\zeta_2\in\mathsf{G}$,
$\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1}(\zeta_1)},[\Re_1(\zeta_2)]_{\alpha_{\Re_1}(\zeta_2)})>0$,
$\Rightarrow[\theta(s(\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1}(\zeta_1)},[\Re_1(\zeta_2)]_{\alpha_{\Re_1(\zeta_2)}})))]\leq[\theta(\mathtt{M}(\zeta_1,\zeta_2))]^k$,
where $\theta\in\Theta_s$ and $\mathtt{M}(\zeta_1,\zeta_2)$ is same as in Corollary 5.2.6.
Then $\Re_1$ has a FP in G.

Corollary 5.2.9. Let $(\mathsf{G}, \mathsf{d}_b)$ be a complete b-MS. Let $\hat{\Re}_1, \hat{\Re}_2 : \mathsf{G} \longrightarrow \mathcal{F}(\mathsf{G})$ be the pair of FMs. If $\hat{\Re}_2(\zeta_1), \hat{\Re}_1(\zeta_2)$ is non empty and $\in \mathfrak{CB}(\mathsf{G})$ such that $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$, $\mathrm{H}(\hat{\Re}_1(\zeta_1), \hat{\Re}_2(\zeta_2)) > 0$, implies

$$\int_0^{[\theta(s\mathrm{H}(\hat{\Re}_1(\zeta_1),\hat{\Re}_2(\zeta_2)))]} \triangle\,(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\theta(\mathtt{M}(\zeta_1,\zeta_2))]^k+\mathfrak{L}\Xi(\zeta_1,\zeta_2)} \triangle\,(\varsigma)\mathrm{d}(\varsigma),$$

where

$$\begin{aligned} &\mathtt{M}(\zeta_1, \zeta_2) \\ &= \max\Bigg\{\mathsf{d}_b(\zeta_1, \hat{\Re}_1(\zeta_1), \mathsf{d}_b(\zeta_2, \hat{\Re}_2(\zeta_2), \mathsf{d}_b(\zeta_1, \zeta_2), \\ &\frac{\{\mathsf{d}_b(\zeta_1, \hat{\Re}_2(\zeta_2) + \mathsf{d}_b(\zeta_2, \hat{\Re}_1(\zeta_1)\}}{2s}\Bigg\}, \end{aligned}$$

and

$$\Xi(\zeta_1, \zeta_2) = \min\Big\{\mathsf{d}_b(\zeta_1, \hat{\Re}_1(\zeta_1), \mathsf{d}_b(\zeta_2, \hat{\Re}_2(\zeta_2), \mathsf{d}_b(\zeta_1, \hat{\Re}_2(\zeta_2), \mathsf{d}_b(\zeta_2, \hat{\Re}_1(\zeta_1)\Big\},$$

also $\theta \in \Theta_s$ and $\mathfrak{L} \geq 0$.

Then $\exists\ \mathrm{f}^* \in \mathsf{G}$ such that $\Re_1(\mathrm{f}^*)(\mathrm{f}^*) \geq \Re_1(\mathrm{f}^*)(\mathrm{f})$ and $\Re_2(\mathrm{f}^*)(\mathrm{f}^*) \geq \Re_2(\mathrm{f}^*)(\mathrm{f})$ for all $\mathrm{f} \in \mathsf{G}$.

Proof. By Theorem 5.2.2 there exists $\mathrm{f}^* \in \mathsf{G}$ so that $\mathrm{f}^* \in \Re_1(\mathrm{f}^*) \cap \Re_2(\mathrm{f}^*)$ with the help of Lemma 3.1.8 $\Re_1(\mathrm{f}^*)(\mathrm{f}^*) \geq \Re_1(\mathrm{f}^*)(\mathrm{f})$ and $\Re_2(\mathrm{f}^*)(\mathrm{f}^*) \geq \Re_1(\mathrm{f}^*)(\mathrm{f})$ for all $\mathrm{f} \in \mathsf{G}$. □

In Corrolary 5.2.9 by taking $\mathfrak{L} = 0$ we obtain:

Corollary 5.2.10. Let $(\mathsf{G}, \mathsf{d}_b)$ be a complete b-MS. Let $\hat{\Re}_1, \hat{\Re}_1 : \mathsf{G} \longrightarrow \mathcal{F}(\mathsf{G})$ be the pair of FMs. If $\hat{\Re}_2(\zeta_1)$ and $\hat{\Re}_1(\zeta_2)$ are non empty and $\in \mathfrak{CB}(\mathsf{G})$ and $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$, $\mathrm{H}(\hat{\Re}_1(\zeta_1), \hat{\Re}_2(\zeta_2)) > 0$, implies

$$\int_0^{[\theta(s\mathrm{H}(\hat{\Re}_1(\zeta_1),\hat{\Re}_2(\zeta_2)))]} \triangle\,(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\theta(\mathtt{M}(\zeta_1,\zeta_2))]^k} \triangle\,(\varsigma)\mathrm{d}(\varsigma),$$

also $\mathtt{M}(\zeta_1, \zeta_2)$ is same as in Corrolary 5.2.9 and $\theta \in \Theta_s$.
Then $\exists\ \mathrm{f}^* \in \mathsf{G}$ such that

$\Re_1(\mathrm{f}^*)(\mathrm{f}^*) \geq \Re_1(\mathrm{f}^*)(\mathrm{f})$ and $\Re_2(\mathrm{f}^*)(\mathrm{f}^*) \geq \Re_1(\mathrm{f}^*)(\mathrm{f})$ for all $\mathrm{f} \in \mathsf{G}$.

If we take $\triangle(\varsigma) = 1$ we get:

Corollary 5.2.11. Let $(\mathsf{G}, \mathsf{d}_b)$ be a complete b-MS. Let $\Re_1, \Re_2 : \mathsf{G} \longrightarrow \mathcal{F}(\mathsf{G})$ be the pair of FMs. If $\hat{\Re}_2(\zeta_1)$ and $\hat{\Re}_1(\zeta_2)$ are none empty and $\in \mathfrak{CB}(\mathsf{G})$, such that $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$,
$\mathrm{H}(\hat{\Re}_1(\zeta_1), \hat{\Re}_2(\zeta_2)) > 0$, implies

$$[\theta(s\mathrm{H}(\hat{\Re}_1(\zeta_1), \hat{\Re}_2(\zeta_2)))] \leq [\theta(\varphi(\mathtt{M}(\zeta_1, \zeta_2)))]^k,$$

where $\mathtt{M}(\zeta_1, \zeta_2)$ is same as in Corrolary 5.2.9 and $\theta \in \Theta_s$.
Then $\exists\ \zeta^* \in \mathsf{G}$ such that $\Re_1(\zeta^*)(\zeta^*) \geq \Re_1(\zeta^*)(\zeta)$ and $\Re_2(\zeta^*)(\zeta^*) \geq \Re_1(\zeta^*)(\zeta)$ for all $\zeta \in \mathsf{G}$.

5.3 Some Consequences

We will briefly discuss some implications of our findings about MMs in this section.

Theorem 5.3.1. Let $\mathsf{G} \neq \emptyset$ be a complete b-MS. Consider the MMs i.e $\mathcal{K}, \mathcal{J} : \mathsf{G} \longrightarrow \mathfrak{CB}(\mathsf{G})$. Suppose that $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$,
$\mathrm{H}(\mathcal{K}(\zeta_1), \mathcal{J}(\zeta_2)) > 0$, implies

$$\int_0^{[\theta(s\mathrm{H}(\mathcal{K}(\zeta_1), \mathcal{J}(\zeta_2)))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\theta(\mathtt{M}(\zeta_1,\zeta_2))]^k + \mathfrak{L}\Xi(\zeta_1,\zeta_2)} \triangle(\varsigma)\mathrm{d}(\varsigma)$$

with,

$$\mathtt{M}(\zeta_1, \zeta_2) = \max\left\{\mathsf{d}_b(\zeta_1, \mathcal{K}(\zeta_1)), \mathsf{d}_b(\zeta_2, \mathcal{J}(\zeta_2)), \mathsf{d}_b(\zeta_1, \zeta_2), \frac{\{\mathsf{d}_b(\zeta_2, \mathcal{J}(\zeta_2)) + \mathsf{d}_b(\zeta_2, \mathcal{K}(\zeta_1))\}}{2s}\right\},$$

and

$$\Xi(\zeta_1,\zeta_2)=\min\Big\{\mathsf{d}_b(\zeta_1,\mathcal{K}(\zeta_1)),\mathsf{d}_b(\zeta_2,\mathcal{J}(\zeta_2)),\mathsf{d}_b(\zeta_1,\mathcal{J}(\zeta_2))+\mathsf{d}_b(\zeta_2,\mathcal{K}(\zeta_1))\Big\},$$

also, $\theta\in\Theta_s$ and $\mathfrak{L}\geq 0$.

Then, $\exists$ f $\in\mathsf{G}$ such that f $\in\{\mathcal{K}(\mathrm{f})\cap\mathcal{J}(\mathrm{f})\}$.

Proof. Take $\alpha:\mathsf{G}\longrightarrow(0,1]$ with $\Re_1,\Re_2$ be the pair of FMs defined by,

$$\Re_1(\zeta_1)(\upsilon)=\begin{cases}\alpha(\upsilon), & \text{if } \upsilon\in\mathcal{K}(\zeta_1),\\ 0, & \text{if } \upsilon\notin\mathcal{K}(\zeta_1),\end{cases}$$

and

$$\Re_2(\zeta_1)(\upsilon)=\begin{cases}\alpha(\upsilon), & \text{if } \upsilon\in\mathcal{J}(\zeta_1),\\ 0, & \text{if } \upsilon\notin\mathcal{J}(\zeta_1).\end{cases}$$

Then,

$$\begin{aligned}[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}&=\Big\{\upsilon:\Re_1(\zeta_1)(\upsilon)\geq\alpha(\zeta_1)\Big\}=\mathcal{K}(\zeta_1),\\ [\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}&=\Big\{\upsilon:\Re_2(\zeta_1)(\upsilon)\geq\alpha(\zeta_1)\Big\}=\mathcal{J}(\zeta_1).\end{aligned}$$

Thus, by Theorem 5.2.2 $\exists$ f $\in\mathsf{G}$ such that f $\in[\Re_1(\mathrm{f})]_{\alpha_{\Re_1(\mathrm{f})}}\cap[\Re_2(\mathrm{f})]_{\alpha_{\Re_2(\mathrm{f})}}=\mathcal{K}(\mathrm{f})\cap\mathcal{J}(\mathrm{f})$. □

If we take $\mathfrak{L}=0$ we obtain:

Corollary 5.3.2. Let $\mathsf{G}\neq\emptyset$ be a complete b-MS. Consider the $\mathcal{K},\mathcal{J}:\mathsf{G}\longrightarrow\mathfrak{CB}(\mathsf{G})$ be a pair of MMs. Suppose that $\forall\ \zeta_1,\zeta_2\in\mathsf{G}$, $\mathrm{H}(\mathcal{K}(\zeta_1),\mathcal{J}(\zeta_2))>0$, implies

$$\int_0^{[\theta(s\mathrm{H}(\mathcal{K}(\zeta_1),\mathcal{J}(\zeta_2)))]}\triangle(\varsigma)\mathrm{d}(\varsigma)\leq\int_0^{[\theta(\mathtt{M}(\zeta_1,\zeta_2))]^k}\triangle(\varsigma)\mathrm{d}(\varsigma)$$

where, $\mathtt{M}(\zeta_1,\zeta_2)$ is same as in Theorem 5.3.1 and $\theta\in\Theta_s$.

Then, $\exists$ f $\in\mathsf{G}$ such that f $\in\{\mathcal{K}(\mathrm{f})\cap\mathcal{J}(\mathrm{f})\}$.

If we take $\triangle(\varsigma)=1$ we get the following:

Corollary 5.3.3. Let $\mathsf{G}\neq\emptyset$ be a complete b-MS. Consider the MMs i.e $\mathcal{K},\mathcal{J}:\mathsf{G}\longrightarrow\mathfrak{CB}(\mathsf{G})$. Suppose that $\forall\ \zeta_1,\zeta_2\in\mathsf{G}$,

$\mathrm{H}(\mathcal{K}(\zeta_1), \mathcal{J}(\zeta_2)) > 0,$

$$\Rightarrow [\theta(s\mathrm{H}(\mathcal{K}(\zeta_1), \mathcal{J}(\zeta_2)))] \leq [\theta(\mathtt{M}(\zeta_1, \zeta_2))]^k,$$

where, $\mathtt{M}(\zeta_1, \zeta_2)$ is same as in Theorem 5.3.1 and $\theta \in \Theta_s$.

Then, $\exists$ f $\in \mathsf{G}$ such that f is a common FP of $\mathcal{K}$ and $\mathcal{J}$.

Theorem 5.3.4. Assume $(\mathsf{G}, \eth)$ being a complete b-metric linear space and let $\Re_1, \Re_2 : \mathsf{G} \to \mathfrak{D}(\mathsf{G})$ be the pair of FMs. Assume there are $\theta \in \Theta_s$ and $\mathfrak{L} \geq 0$ such that $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$, $\eth(\Re_1(\zeta_1), \Re_2(\zeta_2)) > 0$,

$$\Rightarrow \int_0^{[\theta(s\eth(\Re_1(\zeta_1), \Re_2(\zeta_2)))]} \triangle(\varsigma)d(\varsigma) \leq \int_0^{[\theta(\mathtt{M}(\zeta_1, \zeta_2))]^k + \mathfrak{L}\Xi(\zeta_1, \zeta_2)} \triangle(\varsigma)d(\varsigma)$$

with

$$\mathtt{M}(\zeta_1, \zeta_2) = \max\Bigg\{\mathrm{Q}(\zeta_1, \Re_1(\zeta_1)), \mathrm{Q}(\zeta_2, \Re_2(\zeta_2)), \mathrm{Q}(\zeta_1, \zeta_2), \frac{\{\mathrm{Q}(\zeta_1, \Re_2(\zeta_2)) + \mathrm{Q}(\zeta_2, \Re_1(\zeta_1))\}}{2s}\Bigg\},$$

and

$$\Xi(\zeta_1, \zeta_2) = \min\Big\{\mathrm{Q}(\zeta_1, \Re_1(\zeta_1)), \mathrm{Q}(\zeta_2, \Re_2(\zeta_2)), \mathrm{Q}(\zeta_1, \Re_2(\zeta_2)), \mathrm{Q}(\zeta_2, \Re_1(\zeta_1))\Big\},$$

then $\exists$ f $\in \mathsf{G}$ such that $\{\mathrm{f}\} \subset \Re_1(\mathrm{f})$ and $\{\mathrm{f}\} \subset \Re_2(\mathrm{f})$.

Proof. Consider $\zeta_1 \in \mathsf{G}$, by using Lemma 5.1.3 there is $\zeta_2 \in \mathsf{G}$ so that $\zeta_2 \in [\Re_1(\zeta_1)]_1$. Similarly one can obtain $\zeta_3 \in [\Re_2(\zeta_1)]_1$. Then for each $\zeta_1 \in \mathsf{G}$, $[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}$ and $[\Re_2(\zeta_1)]_{\alpha_{\Re_2(\zeta_1)}}$ are non empty and $\in \mathfrak{CB}(\mathsf{G})$. As $\alpha(\zeta_1) = 1$, by the definition of $\eth$ for FSs, we have

$\mathrm{H}([\Re_1(\zeta_1)]_{\alpha(\zeta_1)}, [\Re_2(\zeta_2)]_{\alpha(\zeta_2)}) \leq \eth(\Re_1(\zeta_1), \Re_2(\zeta_2)), \quad \forall\ \zeta_1, \zeta_2 \in \mathsf{G}$. Since θ is non decreasing, we have

$$\int_0^{[\theta(s\mathrm{H}([\Re_1(\zeta_1)]_{\alpha(\zeta_1)},[\Re_2(\zeta_2)]_{\alpha_{(\zeta_2)}}))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\theta(\mathtt{M}(\zeta_1,\zeta_2))]^k+\mathfrak{L}\Xi(\zeta_1,\zeta_2)} \triangle(\varsigma)\mathrm{d}(\varsigma).$$

Since $[\Re_2(\zeta_1)]_1 \subset [\Re_2(\zeta_1)]_{\alpha(\zeta_1)}$,

so $\eth(\zeta_1, [\Re_2(\zeta_1)]_{\alpha(\zeta_1)}) \leq \mathsf{d}_b(\zeta_1, [\Re_2(\zeta_1)]_1)$ for each $\alpha \in (0,1]$.

It yields that

$\mathrm{Q}(\zeta_1, [\Re_1(\zeta_1)]_{\alpha(\zeta_1)}) \leq \mathsf{d}_b(\zeta_1, [\Re_1(\zeta_1)]_1)$.

Similarly,

$\mathrm{Q}(\zeta_1, [\Re_2(\zeta_1)]_{\alpha(\zeta_1)}) \leq \mathsf{d}_b(\zeta_1, [\Re_2(\zeta_1)]_1)$.

$$\int_0^{[\theta(\mathrm{H}([\Re_1(\zeta_1)]_1,[\Re_2(\zeta_2)]_1)]} \triangle(\varsigma)\mathrm{d}(\varsigma) \leq \int_0^{[\theta(\mathtt{M}(\zeta_1,\zeta_2))]^k+\mathfrak{L}\Xi(\zeta_1,\zeta_2)} \triangle(\varsigma)\mathrm{d}(\varsigma).$$

with

$$\begin{aligned}&\mathtt{M}(\zeta_1,\zeta_2) = \\ &\max\Big\{\mathsf{d}_b(\zeta_2,[\Re_2(\zeta_2)]_1), \mathsf{d}_b(\zeta_1,\zeta_2), \mathsf{d}_b(\zeta_1,[\Re_1(\zeta_1)]_1), \\ &\frac{\{\mathsf{d}_b(\zeta_1,[\Re_2(\zeta_2)]_1)+\mathsf{d}_b(\zeta_2,[\Re_1(\zeta_1)]_1)\}}{2s}\Big\},\end{aligned}$$

and

$$\begin{aligned}\Xi(\zeta_1,\zeta_2) = \min\Big\{&\mathsf{d}_b(\zeta_1,[\Re_1(\zeta_1)]_1), \mathsf{d}_b(\zeta_2,[\Re_2(\zeta_2)]_1), \\ &\mathsf{d}_b(\zeta_1,[\Re_2(\zeta_2)]_1), \mathsf{d}_b(\zeta_2,[\Re_1(\zeta_1)]_1)\Big\}.\end{aligned}$$

By Theorem 5.2.2 there is $\mathrm{f} \in \mathsf{G}$ such that $\mathrm{f} \in \{[\Re_1(\mathrm{f})]_1 \cap [\Re_2(\mathrm{f})]_1\}$.

□

By taking $\mathfrak{L} = 0$ in Theorem 5.3.4 leads to the following:

Corollary 5.3.5. Assume $(\mathsf{G}, \eth)$ being a complete b-metric linear space and let $\Re_1, \Re_2 : \mathsf{G} \to \mathfrak{D}(\mathsf{G})$ be the pair of FMs. Assume there is $\theta \in \Theta_s$ such that $\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$,

$\eth(\Re_1(\zeta_1), \Re_2(\zeta_2)) > 0$ implies,

$$\int_0^{[\theta(s\eth(\Re_1(\zeta_1),\Re_2(\zeta_2))]} \triangle(\varsigma)d(\varsigma) \leq \int_0^{[\theta(\mathtt{M}(\zeta_1,\zeta_2))]^k} \triangle(\varsigma)d(\varsigma),$$

where, $\mathtt{M}(\zeta_1, \zeta_2)$ is same as in Theorem 5.3.4 then $\exists$ f $\in \mathsf{G}$ such that $\{f\} \subset \Re_1(f)$ and $\{f\} \subset \Re_2(f)$.

By taking $\triangle(\varsigma) = 0$ in Corollary 5.3.5 leads to the following:

Corollary 5.3.6. Assume $(\mathsf{G}, \eth)$ being a complete b-metric linear space and let $\Re_1, \Re_2 : \mathsf{G} \to \mathfrak{D}(\mathsf{G})$ be the pair of FM. Assume there is $\theta \in \Theta_s$ such that $\forall\, \zeta_1, \zeta_2 \in \mathsf{G}$, $\eth(\Re_1(\zeta_1), \Re_2(\zeta_2)) > 0$ implies,
$[\theta(s\eth(\Re_1(\zeta_1), \Re_2(\zeta_2)] \leq [\theta(\mathtt{M}(\zeta_1, \zeta_2))]^k$,
where, $\mathtt{M}(\zeta_1, \zeta_2)$ is same as in Theorem 5.3.4 then $\exists$ f $\in \mathsf{G}$ a common FP of $\Re_1$ and $\Re_2$.

Remark 5.3.7. If we take $s = 1$, $\mathtt{M} = \mathsf{d}_b(\zeta_1, \zeta_2)$ and $\mathfrak{L} = 0$ then the results of Kanwal et al. [81] becomes special case of our results.

5.4 Application

Here, we investigate the existence of solutions for a system of Volterra stochastic integral equations.

If $\mho$ is a nonempty set, A is an ξ-algebra of subsets of $\mho$, and $\mathtt{P}$ is a complete probability measure on A, then denote by $(\mho, \mathsf{A}, \mathtt{P})$ a probability measure space. Let $\mathbb{R}^+ = [0, \infty)$. The expression $\mathsf{C} = \mathsf{C}(\mathbb{R}_+, \mathcal{L}_2(\mho, \mathsf{A}, \mathtt{P}))$ is used to represent the space of all continuous and bounded functions on $\mathbb{R}^+$ with values in $\mathcal{L}_2 = \mathcal{L}_2(\mho, \mathsf{A}, \mathtt{P})$. Consider the following system of Volterra stochastic integral equation.

$$\zeta_1(\mathsf{t}, \mu) = \int_0^{\mathsf{t}} \mathtt{K}_1(\mathsf{t}, \varrho, \mu)\mathsf{f}(\varrho, \zeta_1(\varrho, \mu))\, d\varrho \; + \beta(\mathsf{t}, \mu), \tag{5.15}$$

$$\zeta_2(\mathsf{t}, \mu) = \int_0^{\mathsf{t}} \mathtt{K}_2(\mathsf{t}, \varrho, \mu)\mathsf{f}(\varrho, \zeta_2(\varrho, \mu))\, d\varrho \; + \beta(\mathsf{t}, \mu), \tag{5.16}$$

where $\mathsf{t} \geq 0$, μ is a point of nonempty set $\mho$, $\beta(\mathsf{t}, \mu)$ is the stochastic free term, $\zeta_1(\mathsf{t}, \mu)$ are unknown stochastic variable, $\mathtt{K}_1, \mathtt{K}_2$ are stochastic kernels defined on $0 \leq \varrho \leq \mathsf{t} < \infty$ and f is a scalar function defined for $\mathsf{t} \geq 0$.
A function that belongs to $\mathsf{C}(\mathbb{R}_+, \mathcal{L}_2(\mho, \mathsf{A}, \mathtt{P}))$ and satisfies the (5.15) and (5.16) is referred to as a random solution.
Define b-MS by

$$\mathsf{d}_b(\zeta_1, \zeta_2) = \max_{\mathsf{t} \in [0,\mathsf{t}]} |\zeta_1(\mathsf{t}) - \zeta_2(\mathsf{t})|^2,$$

$\forall\ \zeta_1, \zeta_2 \in \mathsf{C}[0, \mathsf{t}]$. Then $(\mathsf{C}[0, \mathsf{t}], \mathsf{d}_b)$ is a complete b-MS with $s = 2$.

Theorem 5.4.1. Consider the (5.15) and (5.16) Volterra stochastic integral Equation system defined above with the following assumptions:

(i): $\mathsf{f} : \mathsf{C} \longrightarrow \mathsf{C},\ \beta : \mathbb{R}_+ \longrightarrow \mathcal{L}_2, \mathtt{K}_1, \mathtt{K}_2 : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{L}_2 \longrightarrow \mathcal{L}_2$ are continuous.

(ii): $\forall\ \mathtt{K}_1, \mathtt{K}_2 \in \mathtt{K}$ and $\lambda \in (0, 1)$. we have,

$$||\mathtt{K}_1(\mathsf{t}, \varrho, \mu) - \mathtt{K}_2(\mathsf{t}, \varrho, \mu)|| \leq (\lambda)\mathtt{K}$$

(iii):

$$\begin{aligned} ||\mathsf{f}(\mathsf{t}, \zeta_1(\mathsf{t}, \mu)) - \mathsf{f}(\mathsf{t}, \zeta_2(\mathsf{t}, \mu))|| &\leq ||\zeta_1(\mathsf{t}, \mu) - \zeta_2(\mathsf{t}, \mu)|| \\ &\leq \max_{\mathsf{t} \in [0,\mathsf{t}]} |\zeta_1(\mathsf{t}, \mu) - \zeta_2(\mathsf{t}, \mu)|^2 \end{aligned}$$

By assumption (i)-(iii), the integral Equations (5.15) and (5.16) have a common solution in $\mathsf{C}[0, \mathsf{t}]$.

Proof. Consider $\mathsf{G} = \mathsf{C} = \mathsf{C}(\mathbb{R}_+, \mathcal{L}_2(\mho, \mathsf{A}, \mathtt{P}))$ be endowed with uniform norm $||.||$. Then , $(\mathsf{G}, ||.||)$ is a Banach space. Let $\mathcal{P}_{(\zeta_1)}, \mathsf{E}_{(\zeta_1)} \in \mathsf{G}$, where

$$\mathcal{P}_{(\zeta_1)} = \int_0^{\mathsf{t}} \mathtt{K}_1(\mathsf{t}, \varrho, \mu)\mathsf{f}(\varrho, \zeta_1(\varrho, \mu))\, d\varrho\ + \beta(\mathsf{t}, \mu),$$

$$\mathtt{E}_{(\zeta_1)} = \int_0^{\mathtt{t}} \mathtt{K}_2(\mathtt{t},\varrho,\mu)\mathtt{f}(\varrho,\zeta_1(\varrho,\mu))\,d\varrho \;+\beta(\mathtt{t},\mu).$$

Assume two arbitrary mappings $\mathcal{A},\mathcal{B} : \mathsf{G} \longrightarrow (0,1]$ and a pair of FM i.e $\Re_1, \Re_2 : \mathsf{G} \longrightarrow \mathcal{F}(\mathsf{G})$ by,

$$\Re_1(\zeta_1)(\mathtt{t}) = \begin{cases} \mathcal{A}(\zeta_1), & \text{if } \zeta_1(\mathtt{t}) = \mathcal{P}_{(\zeta_1)}, \\ 0, & \textit{otherwise}. \end{cases}$$

Also,

$$\Re_2(\zeta_1)(\mathtt{t}) = \begin{cases} \mathcal{B}(\zeta_1), & \text{if } \zeta_1(\mathtt{t}) = \mathtt{E}_{(\zeta_1)}, \\ 0, & \textit{otherwise}. \end{cases}$$

If we take $\alpha_{\Re_{1\zeta_1}} = \mathcal{A}(\zeta_1)$ and $\alpha_{\Re_{2\zeta_1}} = \mathcal{B}(\zeta_1)$. Then, we have

$$[\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}} = \Big\{\mathtt{t} : \Re_1(\zeta_1)(\mathtt{t}) \geq \mathcal{A}(\zeta_1)\Big\} = \mathcal{P}_{(\zeta_1)}.$$
$$[\Re_2(\zeta_1)]_{\alpha_{\Re_2}(\zeta_1)} = \Big\{\mathtt{t} : \Re_2(\zeta_1)(\mathtt{t}) \geq \mathcal{B}\zeta_1)\Big\} = \mathtt{E}_{(\zeta_1)}.$$

Now

$\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}}, [\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}) = ||\mathcal{P}_{(\zeta_1)} - \mathtt{E}_{(\zeta_1)}||.$

$$\begin{aligned}
&\Rightarrow \int_0^{[\theta(s\mathrm{H}([\Re_1(\zeta_1)]_{\alpha_{\Re_1(\zeta_1)}},[\Re_2(\zeta_2)]_{\alpha_{\Re_2(\zeta_2)}}))]} \triangle(\varsigma)\mathrm{d}(\varsigma) \\
&= \int_0^{[\theta(||\mathcal{P}_{(\zeta_1)}-\mathtt{E}_{(\zeta_1)}||)]} \triangle(\varsigma)\mathrm{d}(\varsigma) \\
&= \int_0^{[\theta(||(\int_0^{\mathtt{t}}\mathtt{K}_1(\mathtt{t},\varrho,\mu)\mathtt{f}(\varrho,\zeta_1(\varrho,\mu))\,d\varrho\;+\beta(\mathtt{t},\mu))-(\int_0^{\mathtt{t}}\mathtt{K}_2(\mathtt{t},\varrho,\mu)\mathtt{f}(\varrho,\zeta_2(\varrho,\mu))\,d\varrho\;+\beta(\mathtt{t},\mu))||)]} \triangle(\varsigma)\mathrm{d}(\varsigma). \\
&\leq \int_0^{[\theta(||\int_0^{\mathtt{t}}\mathtt{K}_1(\mathtt{t},\varrho,\mu)\,d\varrho\;-\int_0^{\mathtt{t}}\mathtt{K}_2(\mathtt{t},\varrho,\mu)\,d\varrho\;||||\mathtt{f}(\varrho,\zeta_1(\varrho,\mu))-\mathtt{f}(\varrho,\zeta_2(\varrho,\mu))||)]} \triangle(\varsigma)\mathrm{d}(\varsigma) \\
&\leq \int_0^{[\theta(\lambda\mathtt{K}||\zeta_1(\mathtt{t},\mu)-\zeta_2(\mathtt{t},\mu)||)]^{\frac{1}{v}}} \triangle(\varsigma)\mathrm{d}(\varsigma) \\
&\leq \int_0^{[\theta(||\zeta_1(\mathtt{t},\mu)-\zeta_2(\mathtt{t},\mu)||)]^{\frac{1}{v}}} \triangle(\varsigma)\mathrm{d}(\varsigma) \\
&\leq \int_0^{[\theta([\max_{\mathtt{t}\in[0,a]}|\zeta_1(\mathtt{t})-\zeta_2(\mathtt{t})|^2)]^{\frac{1}{v}}} \triangle(\varsigma)\mathrm{d}(\varsigma) \\
&\leq \int_0^{[\theta(\mathsf{d}_b(\zeta_1,\zeta_2))]^{\frac{1}{v}}} \triangle(\varsigma)\mathrm{d}(\varsigma),
\end{aligned}$$

where $\frac{1}{v} = k \in (0, 1)$. So, all the conditions of Theorem 5.2.2 are satisfied. Hence, integral inclusion (5.15) and (5.16) have a common solution. □

5.5 Conclusion

This chapter is about the existence of common FPs for a pair of FMs and MMs in the framework of complete b-MS. To achieve the goal, a generalized integral-type contraction is employed. Some corollaries are provided to guarantee that our results extend many previously proven classical results.

Chapter 6

FP Results of an Extended Contractions

In this chapter we presents new common FP theorems for multivalued operators, introducing innovative contraction conditions based on rational advanced Nashine-Wardowski-Feng-Liu type contractions within the framework of orbitally complete controlled MSs. By integrating generalized contraction principles and order structures, we develop a flexible approach that extends and unifies existing results.

6.1 Chapter Layout

To promote a better understanding of the material presented in this chapter, the following organization has been adopted:

(i): The chapter opens with a set of fundamental definitions that lay the groundwork for the concepts and results discussed in the subsequent sections.

(ii): Section 6.2 presents the introduction of the set $\nabla \mathtt{F}^*{}_{\sigma}$ within orbitally complete controlled MSs, followed by the establishment of a FP theorem. Additionally, a FP result is derived in the context of ordered controlled MSs, supported by a relevant example for validation.

(iii): In Section 6.3, we explore the existing research gap and propose a generalization to further substantiate the developed results.

(iv): Section 6.4 focuses on the application of the main findings to prove the existence of solutions to a class of nonlinear integral equations, along with graphical illustrations to support the analysis.

(v): In Section 6.5, the established theoretical results are utilized to demonstrate the existence of solutions to fractional differential equations.

(vi): Finally, Section 6.6 offers concluding remarks of the key contributions, providing a comprehensive wrap-up of the chapter.

Throughout the chapter, $\mathsf{P}(\mathsf{G})$ stands for the set of all closed and compact subsets of G.

Definition 6.1.1. Let $\alpha : \mathsf{G}^2 \longrightarrow [0, +\infty)$. A mapping $\Re_1 : \mathsf{G} \longrightarrow \mathsf{P}(\mathsf{G})$ fulfilling:

$$\alpha_*(\Re_1\zeta_1, \Re_1\zeta_2) = \inf\{\alpha(\zeta_3, \zeta_4) : \zeta_3 \in \Re_1\zeta_1, \zeta_4 \in \Re_1\zeta_2\} \geq 1,$$

whenever $\alpha(\zeta_3, \zeta_4) \geq 1, \ \forall\ \zeta_3, \zeta_4 \in \mathsf{G}$ is called α_*-admissible.
A mapping $\Re_1 : \mathsf{G} \longrightarrow \mathsf{P}(\mathsf{G})$ with:

$$\alpha_*(\zeta_1, \Re_1(\zeta_1)) = \inf\{\alpha(\zeta_1, \zeta_2) : \zeta_2 \in \Re_1(\zeta_1)\} \geq 1,$$

$\forall\ \zeta_1, \zeta_2 \in \mathsf{G}$ is called α_*-dominated on K, where $\mathsf{K} \subseteq \mathsf{G}$. [107]

Definition 6.1.2. Let $\Re_1, \Re_2 : \mathsf{G} \longrightarrow \mathsf{G}$, for any $\zeta_0 \in \mathsf{G}, \mathsf{O}(\zeta_0) = \{\zeta_0, \Re_1(\zeta_0), \Re_2(\zeta_1)\}$ indicates the orbit of ζ_0. Consider a mapping $\mathsf{f} : \mathsf{G} \longrightarrow \mathbb{R}$ then it is said to be $(\Re_1, \Re_2)$ orbitally LSC if $\mathsf{f}(\zeta) < \lim_{j\longrightarrow\infty} \inf \mathsf{f}(\zeta_j)$ for all sequences $\{\Re_1\Re_2(\zeta_j) \subset \mathsf{O}(\zeta_0)\}$ with $\lim_{j\longrightarrow\infty} \{\Re_1\Re_2(\zeta_j)\} = \zeta \in \mathsf{G}$.

Example 6.1.3. Consider a set $\mathsf{G} = [-2, 2]$ and a self map $\Re_1 \colon \mathsf{G} \to \mathsf{G}$ defined as:

$$\Re_1(\zeta) = \frac{\zeta}{2}.$$

For $\zeta_0 \in (0,2)$, the orbit of ζ_0 with respect to $\Re_1$ is given by

$$\mathtt{O}_{\Re_1}(\zeta_0) = \{\zeta_0, \frac{\zeta_0}{2}, \frac{\zeta_0}{4}, ...\}.$$

Let $\{\zeta_t\}$ be any sequence in $\mathtt{O}_{\Re_1}(\zeta_0)$ which converges to zero. Now consider a function $\Re_2\colon \mathsf{G} \to \mathbb{R}$ defined by $\Re_2(\zeta) = |\zeta|$. One can easily check that

$$\begin{aligned}\lim_{t\to\infty} \inf \Re_2(\zeta_t) &= 0\\ &= \Re_2(0).\end{aligned}$$

Hence, $\Re_2$ is $\Re_1$-orbitally LSC at $\zeta = 0$.

Example 6.1.4. Consider the set $\mathsf{G} = \{0,1,2\}$ and the function d_σ defined as:

$$\begin{aligned}&\mathsf{d}_\sigma(0,0) = \mathsf{d}_\sigma(1,1) = \mathsf{d}_\sigma(2,2) = 0,\\ &\mathsf{d}_\sigma(0,1) = \mathsf{d}_\sigma(1,0) = 1,\\ &\mathsf{d}_\sigma(0,2) = \mathsf{d}_\sigma(2,0) = \frac{3}{2},\\ &\mathsf{d}_\sigma(1,2) = \mathsf{d}_\sigma(2,1) = \frac{2}{5}.\end{aligned}$$

Let $\sigma : \mathsf{G} \times \mathsf{G} \to [1,\infty)$ be a symmetric function defined by:

$$\begin{aligned}&\sigma(0,0) = \sigma(1,1) = \sigma(2,2) = \sigma(0,2) = 1,\\ &\sigma(1,2) = \frac{5}{4},\\ &\sigma(0,1) = \frac{11}{10}.\end{aligned}$$

It is straightforward to verify that d_σ satisfies the conditions of a controlled metric. Furthermore, observe that:

$$\begin{aligned}\mathsf{d}_\sigma(0,1) &\leq s[\mathsf{d}_\sigma(0,2) + \mathsf{d}_\sigma(2,1)]\\ 1 &\leq s\left[\frac{3}{2} + \frac{2}{5}\right]\\ 1 &\leq s\left[\frac{19}{10}\right]\\ s &\geq \frac{10}{19}\end{aligned}$$

Thus, d_σ is not a b-metric for the same function $\sigma(\zeta_1, \zeta_3) = s \ \forall \ \zeta_1, \zeta_3 \in \mathsf{G}$.

Definition 6.1.5. Let $\Re_1, \Re_2 : \mathsf{G} \longrightarrow \mathrm{P}(\mathsf{G})$ be two MM. An orbit for a pair $(\Re_1, \Re_2)$ at a point $\zeta_0 \in \mathsf{G}$ denoted by $\mathtt{O}(\zeta_0)$ is a sequence defined as:

$$\{\zeta_j : \zeta_j \in \Re_1\Re_2(\zeta_{j-1})\}.$$

Definition 6.1.6. Let $\Re_1, \Re_2 : \mathsf{G} \longrightarrow \mathrm{P}(\mathsf{G})$ be two MM. If a Cauchy sequence $\{\zeta_j : \zeta_j \in \Re_1\Re_2(\zeta_{j-1})\}$ converges in complete controlled MSs, then G is said to be $(\Re_1, \Re_2)$ orbitally complete.

6.2 FP Results in Orbitally Controlled MS

Definition 6.2.1. Let $\Re_1, \Re_2 : \mathsf{G} \longrightarrow \mathrm{P}(\mathsf{G})$ be two MM, $\mathtt{F} \in \nabla\mathtt{F}^*{}_\sigma$ and $\nu : (0, \infty) \longrightarrow (0, \infty)$. For all $\zeta_1, \zeta_2 \in \mathsf{G}$ with $\max\{\mathsf{d}_\sigma(\zeta_1, \Re_1(\zeta_1)), \mathsf{d}_\sigma(\zeta_2, \Re_2(\zeta_2))\} > 0$, define a set $\mathtt{F}_\nu^{\zeta_1} \subset \mathsf{G}$ as

$$\mathtt{F}_\nu^{\zeta_1} = \left\{ \begin{array}{c} \zeta_2 \in \Re_1(\zeta_1), \zeta_3 \in \Re_2(\zeta_2) : \mathtt{F}(\mathsf{d}_\sigma(\zeta_2, \zeta_3)) \\ \leq \mathtt{F}(\mathtt{M}(\zeta_1, \zeta_2)) + \mathcal{L}\Xi(\zeta_1, \zeta_2) + \nu(\mathsf{d}_\sigma(\zeta_2, \zeta_3)) \end{array} \right\}, \quad \forall \ \mathcal{L} \geq 0$$

where

$$\mathtt{M}(\zeta_1, \zeta_2) = \max\left\{ \mathsf{d}_\sigma(\zeta_1, \Re_1(\zeta_1)), \mathsf{d}_\sigma(\zeta_2, \Re_2(\zeta_2)), \frac{\{\mathsf{d}_\sigma(\zeta_1, \Re_1(\zeta_1)) \times \mathsf{d}_\sigma(\zeta_2, \Re_2(\zeta_2))\}}{1 + \mathsf{d}_\sigma(\zeta_1, \zeta_2)} \right\},$$

and

$$\Xi(\zeta_1, \zeta_2) = \min\left\{ \mathsf{d}_\sigma(\zeta_1, \Re_1(\zeta_1)), \mathsf{d}_\sigma(\zeta_2, \Re_2(\zeta_2)), \mathsf{d}_\sigma(\zeta_1, \Re_2(\zeta_2)), \mathsf{d}_\sigma(\zeta_2, \Re_1(\zeta_1)) \right\}.$$

Theorem 6.2.2. Let $(\mathsf{G}, \mathsf{d}_\sigma)$ be an orbitally complete controlled MS. Let $\zeta_0 \in \mathsf{G}, \alpha : \mathsf{G}^2 \longrightarrow [0, \infty)$ and $\Re_1, \Re_2 : \mathsf{G} \longrightarrow \mathrm{P}(\mathsf{G})$ be two α_*-dominated MMs and $\mathtt{F} \in \nabla\mathtt{F}^*{}_\sigma$. Assume the following hold:

(i):

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\sigma(\zeta_{i+1}, \zeta_{i+2})\sigma(\zeta_{i+1}, \zeta_m)}{\sigma(\zeta_i, \zeta_{i+1})} < \frac{1}{\mathtt{P}} \qquad \forall \ \mathtt{P} \in (0, 1); \tag{6.1}$$

(ii): the mapping $\zeta_3 \mapsto \max\{\mathsf{d}_\sigma(\zeta_1, \Re_1(\zeta_1)), \mathsf{d}_\sigma(\zeta_2, \Re_2(\zeta_2))\}$ is orbitally LSC;

(iii): $\exists$ functions $\tau, \nu : (0, \infty) \longrightarrow (0, \infty)$ such that

$$\tau(\mathsf{t}) > \nu(\mathsf{t}), \qquad \lim_{\mathsf{s}\longrightarrow \mathsf{t}^+} \inf \tau(\mathsf{s}) > \lim_{\mathsf{s}\longrightarrow \mathsf{t}^+} \inf \nu(\mathsf{s}) \text{ for all } \mathsf{t} \in (0, \infty)$$

(iv): $\forall\ \zeta_1, \zeta_2 \in \{\Re_1\Re_2(\zeta_j) \subset \mathtt{O}(\zeta_0)\}$ with $\alpha(\zeta_1, \zeta_2) \geq 1$ and

$\max\{\mathsf{d}_\sigma(\zeta_1, \Re_1(\zeta_1)), \mathsf{d}_\sigma(\zeta_2, \Re_2(\zeta_2))\} > 0$, there exist $\zeta_1, \zeta_2 \in \mathtt{F}^{\zeta_1}{}_\sigma$ satisfying;

$$\tau(\mathsf{d}_\sigma(\zeta_1, \zeta_2)) + \mathtt{F}(\mathtt{M}(\zeta_1, \zeta_2)) + \mathcal{L}\Xi(\zeta_1, \zeta_2) \leq \mathtt{F}(\mathsf{d}_\sigma(\zeta_1, \zeta_2)) \qquad \forall\ \mathcal{L} \geq 0. \tag{6.2}$$

Then $\Re_1$ and $\Re_2$ possess a common FP.

Proof. Assume $\zeta_{2j+1} \in \Re_1(\zeta_{2j})$ and $\zeta_{2j+2} \in \Re_2(\zeta_{2j+1})$. We construct a sequence $\{\Re_1\Re_2(\zeta_j)\} \in \mathsf{G}$ for every $j \in \mathbb{N} \cup \{0\}$.

Furthermore, $\mathsf{d}_\sigma(\zeta_{2j}, \Re_1(\zeta_{2j})) = \mathsf{d}_\sigma(\zeta_{2j}, \zeta_{2j+1})$ for some $\zeta_{2j+1} \in \Re_1(\zeta_{2j})$ and $\mathsf{d}_\sigma(\zeta_{2j+1}, \Re_2(\zeta_{2j+1})) = \mathsf{d}_\sigma(\zeta_{2j+1}, \zeta_{2j+2})$ for some $\zeta_{2j+2} \in \Re_2(\zeta_{2j+1})$, implies $\{\Re_1\Re_2(\zeta_j)\}$ is a sequence in G generated by ζ_0.

Assume that $\Re_1$ and $\Re_2$ have no FP. Then we have $\max\{\mathsf{d}_\sigma(\zeta_1, \Re_1(\zeta_1)), \mathsf{d}_\sigma(\zeta_2, \Re_2(\zeta_2))\} > 0$ for each $\zeta_1, \zeta_2 \in \mathsf{G}$.

Since $\Re_1(\zeta_1), \Re_2(\zeta_2) \in \mathrm{P}(\mathsf{G})\ \forall\ \zeta_1, \zeta_2 \in \mathsf{G}$ and $\mathtt{F} \in \nabla\mathtt{F}^*{}_\sigma$, it is easy to show that $\mathtt{F}^{\zeta_1}{}_\sigma \neq \emptyset$.

As $\Re_1$ and $\Re_2$ are two α_*-dominated MMs and $\alpha_*(\zeta_{2j}, \Re_1(\zeta_{2j})) \geq 1$, $\Rightarrow$ $\inf\{\alpha(\zeta_{2j}, b) : b \in \Re_1(\zeta_{2j})\} \geq 1$.

Therefore,

$\alpha(\zeta_{2j}, \zeta_{2j+1}) \geq 1$. If $\zeta_0 \in \mathsf{G}$ is any initial point, then $\zeta_{2j}, \zeta_{2j+1} \in \mathtt{F}^{\zeta_0}{}_\sigma$ and by using (6.2), we obtain

$$\tau(\mathsf{d}_\sigma(\zeta_{2j}, \zeta_{2j+1})) + \mathtt{F}(\mathtt{M}(\zeta_{2j}, \zeta_{2j+1})) + \mathcal{L}\Xi(\zeta_{2j}, \zeta_{2j+1}) \leq \mathtt{F}(\mathsf{d}_\sigma(\zeta_{2j}, \zeta_{2j+1})). \tag{6.3}$$

Here,

$$\begin{aligned}
&\mathtt{M}(\zeta_{2j}, \zeta_{2j+1}) \\
&= \max\left\{ \mathsf{d}_\sigma(\zeta_{2j}, \Re_1(\zeta_{2j})), \mathsf{d}_\sigma(\zeta_{2j+1}, \Re_2(\zeta_{2j+1})), \right. \\
&\qquad \left. \frac{\{\mathsf{d}_\sigma(\zeta_{2j}, \Re_1(\zeta_{2j})) \times \mathsf{d}_\sigma(\zeta_{2j+1}, \Re_2(\zeta_{2j+1}))\}}{1 + \mathsf{d}_\sigma(\zeta_{2j}, \zeta_{2j+1})} \right\} \\
&\leq \max\left\{ \mathsf{d}_\sigma(\zeta_{2j}, \zeta_{2j+1}), \mathsf{d}_\sigma(\zeta_{2j+1}, \zeta_{2j+2}), \frac{\{\mathsf{d}_\sigma(\zeta_{2j}, \zeta_{2j+1}) \times \mathsf{d}_\sigma(\zeta_{2j+1}, \zeta_{2j+2})\}}{1 + \mathsf{d}_\sigma(\zeta_{2j}, \zeta_{2j+1})} \right\} \\
&= \max\left\{ \mathsf{d}_\sigma(\zeta_{2j}, \zeta_{2j+1}), \mathsf{d}_\sigma(\zeta_{2j+1}, \zeta_{2j+2}) \right\},
\end{aligned}$$

and

$$\begin{aligned}
&\Xi(\zeta_{2j}, \zeta_{2j+1}) \\
&= \min\left\{ \mathsf{d}_\sigma(\zeta_{2j}, \zeta_{2j+1}), \mathsf{d}_\sigma(\zeta_{2j+1}, \zeta_{2j+2}), \mathsf{d}_\sigma(\zeta_{2j}, \zeta_{2j+2}), \mathsf{d}_\sigma(\zeta_{2j+1}, \zeta_{2j+1}) \right\} \\
&= 0.
\end{aligned}$$

If we take

$$\max\left\{ \mathsf{d}_\sigma(\zeta_{2j}, \zeta_{2j+1}), \mathsf{d}_\sigma(\zeta_{2j+1}, \zeta_{2j+2}) \right\} = \mathsf{d}_\sigma(\zeta_{2j}, \zeta_{2j+1}),$$

then (6.3) becomes

$$\tau(\mathsf{d}_\sigma(\zeta_{2j}, \zeta_{2j+1})) + \mathtt{F}(\mathsf{d}_\sigma(\zeta_{2j}, \zeta_{2j+1})) \leq \mathtt{F}(\mathsf{d}_\sigma(\zeta_{2j}, \zeta_{2j+1})),$$

which is a contradiction. Therefore,

$$\max\left\{ \mathsf{d}_\sigma(\zeta_{2j}, \zeta_{2j+1}), \mathsf{d}_\sigma(\zeta_{2j+1}, \zeta_{2j+2}) \right\} = \mathsf{d}_\sigma(\zeta_{2j+1}, \zeta_{2j+2}).$$

From (6.3)

$$\tau(\mathsf{d}_\sigma(\zeta_{2j}, \zeta_{2j+1})) + \mathtt{F}(\mathsf{d}_\sigma(\zeta_{2j+1}, \zeta_{2j+2})) \leq \mathtt{F}(\mathsf{d}_\sigma(\zeta_{2j}, \zeta_{2j+1})).$$

$$\mathtt{F}(\mathtt{d}_\sigma(\zeta_{2j+1},\zeta_{2j+2})) \leq \mathtt{F}(\mathtt{d}_\sigma(\zeta_{2j},\zeta_{2j+1})) - \tau(\mathtt{d}_\sigma(\zeta_{2j},\zeta_{2j+1})). \tag{6.4}$$

Since $\zeta_{2j+1} \in \mathtt{F}_\nu^{\zeta_1}$ then by definition of $\mathtt{F}_\nu^{\zeta_1}$ we have,

$$\begin{aligned}\mathtt{F}(\mathtt{d}(\zeta_{2j},\zeta_{2j+1})) &\leq \mathtt{F}(\mathtt{M}(\zeta_{2j-1},\zeta_{2j})) + \mathcal{L}\Xi(\zeta_{2j-1},\zeta_{2j}) + \nu(\mathtt{d}_\sigma(\zeta_{2j},\zeta_{2j+1})) \\ &\leq \mathtt{F}(\mathtt{d}_\sigma(\zeta_{2j},\zeta_{2j+1})) + \nu(\mathtt{d}_\sigma(\zeta_{2j},\zeta_{2j+1})).\end{aligned} \tag{6.5}$$

Using (6.5) in (6.4), we obtain

$$\mathtt{F}(\mathtt{d}_\sigma(\zeta_{2j+1},\zeta_{2j+2})) \leq \mathtt{F}(\mathtt{d}_\sigma(\zeta_{2j},\zeta_{2j+1})) + \nu(\mathtt{d}_\sigma(\zeta_{2j},\zeta_{2j+1})) - \tau(\mathtt{d}_\sigma(\zeta_{2j},\zeta_{2j+1})), \tag{6.6}$$

and

$$\mathtt{F}(\mathtt{d}_\sigma(\zeta_{2j},\zeta_{2j+1})) \leq \mathtt{F}(\mathtt{d}_\sigma(\zeta_{2j-1},\zeta_{2j})) + \nu(\mathtt{d}_\sigma(\zeta_{2j-1},\zeta_{2j})) - \tau(\mathtt{d}_\sigma(\zeta_{2j-1},\zeta_{2j})). \tag{6.7}$$

So from (6.6) and (6.7), one gets

$$\begin{aligned}&\mathtt{F}(\mathtt{d}_\sigma(\zeta_{2j+1},\zeta_{2j+2})) \\ &\leq \mathtt{F}(\mathtt{d}_\sigma(\zeta_{2j-1},\zeta_{2j})) + \nu(\mathtt{d}_\sigma(\zeta_{2j},\zeta_{2j+1})) \\ &\quad + \nu(\mathtt{d}_\sigma(\zeta_{2j-1},\zeta_{2j})) - \tau(\mathtt{d}_\sigma(\zeta_{2j},\zeta_{2j+1})) - \tau(\mathtt{d}_\sigma(\zeta_{2j-1},\zeta_{2j})).\end{aligned}$$

Since $\Gamma(\mathtt{d}_\sigma(\zeta_{2j},\zeta_{2j+1})) = \tau(\mathtt{d}_\sigma(\zeta_{2j},\zeta_{2j+1})) - \nu(\mathtt{d}_\sigma(\zeta_{2j},\zeta_{2j+1})) \geq 0$, we have

$$\begin{aligned}&\mathtt{F}(\mathtt{d}_\sigma(\zeta_{2j+1},\zeta_{2j+2})) \\ &\leq \mathtt{F}(\mathtt{d}_\sigma(\zeta_{2j},\zeta_{2j+1})) - \Gamma(\mathtt{d}_\sigma(\zeta_{2j},\zeta_{2j+1})) < \mathtt{F}(\mathtt{d}_\sigma(\zeta_{2j},\zeta_{2j+1})).\end{aligned} \tag{6.8}$$

Now let $\mathtt{d}_\sigma(\zeta_{2j},\zeta_{2j+1}) = \Omega_j$ for all $j \in \mathbb{N} \cup \{0\}$, $\Omega_j > 0$. From (6.8), we have that $\{\Omega_j\}$ is decreasing. Therefore, $\exists\ \delta > 0$ such that $\lim_{j\to\infty} \Omega_j = \delta$. Then using the previous inequality, we have

$$\begin{aligned}\mathtt{F}(\Omega_{j+1}) &\leq \mathtt{F}(\Omega_j) - \Gamma(\Omega_j) \\ &\leq \mathtt{F}(\Omega_{j-1}) - \Gamma(\Omega_j) - \Gamma(\Omega_{j-1}) \\ &\ \vdots \\ &\leq \mathtt{F}(\Omega_0) - \Gamma(\Omega_j) - \Gamma(\Omega_{j-1}) \cdots \Gamma(\Omega_0).\end{aligned}$$

Let w_j be the greatest number in $\{0, 1, \ldots, j-1\}$ such that

$$\Gamma(\Omega_{\mathsf{w}_j}) = \min\{\Gamma(\Omega_0), \Gamma(\Omega_1), \ldots, \Gamma(\Omega_j)\}$$

for all $j \in \mathbb{N} \cup \{0\}$. In this case, $\{\mathsf{w}_j\}$ is a nondecreasing sequence. Thus we have

$$\mathtt{F}(\Omega_j) \leq \mathtt{F}(\Omega_0) - j\Gamma(\Omega_{\mathsf{w}_j}). \tag{6.9}$$

Now, consider the sequence $\{\Gamma(\Omega_{\mathsf{w}_j})\}$. We will discuss two cases:
Case 1: For each $j \in \mathbb{N}$ there is $\mathfrak{s} > j$ such that

$$\Gamma(\Omega_{\mathsf{w}_j}) > \Gamma(\Omega_{\mathsf{w}_\mathfrak{s}}).$$

Then we obtain a subsequence $\{\Omega_{\mathsf{w}_{j_k}}\}$ of $\{\Omega_{\mathsf{w}_j}\}$ with $\Gamma(\Omega_{\mathsf{w}_{j_k}}) > \Gamma(\Omega_{\mathsf{w}_{j_{k+1}}})\ \forall\ k$. Since $\Omega_{\mathsf{w}_{j_k}} \to \delta$ we deduce that

$$\lim_{k\to\infty} \inf \Gamma(\Omega_{\mathsf{w}_{j_k}}) > 0.$$

Hence

$$\mathtt{F}(\Omega_{\mathsf{w}_{j_k}}) \leq \mathtt{F}(\Omega_0) - j^k\Gamma(\Omega_{\mathsf{w}_{j_k}}) \text{ for all } k.$$

Consequently, $\lim_{k\to\infty} \mathtt{F}(\Omega_{\mathsf{w}_{j_k}}) = -\infty$ and by $(\mathcal{F}_2)$, $\lim_{k\to\infty} \Omega_{\mathsf{w}_{j_k}} = 0$ which contradicts the fact that

$$\lim_{k\to\infty} \Omega_{\mathsf{w}_{j_k}} > 0.$$

Case 2: There is $j_0 \in \mathbb{N}$ such that $\Gamma(\Omega_{j_0}) > \Gamma(\Omega_{j_\mathfrak{s}})\ \ \forall\ \mathfrak{s} > j_0$. Then

$$\mathtt{F}(\Omega_\mathfrak{s}) \leq \mathtt{F}(\Omega_0) - \mathfrak{s}\Gamma(\Omega_{\mathsf{w}_{j_0}})$$

for all $\mathfrak{s} > j_0$. Hence, $\lim_{\mathfrak{s}\to\infty} \mathtt{F}(\Omega_\mathfrak{s}) = -\infty$ and by $(\mathcal{F}_2)$, $\lim_{\mathfrak{s}\to\infty} \Omega_\mathfrak{s} = 0$, which contradicts the fact that $\lim_{\mathfrak{s}\to\infty} \Omega_\mathfrak{s} > 0$.
Therefore, in both cases,

$$\lim_{j\to\infty} \Omega_j = 0.$$

Now, from $(\mathcal{F}_3)$, there exists $k \in (0,1)$ such that $\lim_{j\to\infty} (\Omega_j)^k \mathtt{F}(\Omega_j) = 0$.
By (6.9), the following holds $\forall\ j \in \mathbb{N} \cup \{0\}$:

$$(\Omega_j)^k \mathtt{F}(\Omega_j) - (\Omega_j)^k \mathtt{F}(\Omega_0) \leq (\Omega_j)^k (\mathtt{F}(\Omega_j) - j\Gamma(\Omega_{\mathsf{w}_j})) - (\Omega_j)^k (\mathtt{F}(\Omega_0)$$
$$= -j(\Omega_j)^k \Gamma(\Omega_{\mathsf{w}_j})$$
$$\leq 0.$$

Taking limit as $j \to \infty$, we obtain

$$\lim_{j\to\infty} j(\Omega_j)^k \Gamma(\Omega_{\mathsf{w}_j}) = 0$$

Since $\aleph = \lim_{j\to\infty}\inf \Gamma(\Omega_{\mathsf{w}_j}) > 0$, there exists $j_0 \in \mathbb{N} \cup \{0\}$ such that $\Gamma(\Omega_{\mathsf{w}_j}) > \frac{\aleph}{2}$ for all $j \geq j_0$.
Thus,

$$j(\Omega_j)^k \frac{\aleph}{2} < j(\Omega_j)^k \Gamma(\Omega_{\mathsf{w}_j}) \quad \text{for all } j \geq j_0. \tag{6.10}$$
$$\Rightarrow 0 \leq \lim_{j\to\infty} j(\Omega_j)^k \frac{\aleph}{2} < \lim_{j\to\infty} j(\Omega_j)^k \Gamma(\Omega_{\mathsf{w}_j})$$
$$= 0,$$
$$\Rightarrow \lim_{j\to\infty} j(\Omega_j)^k = 0. \tag{6.11}$$

From (6.11), $\exists\ j_1 \in \mathbb{N}$ such that $j(\Omega_j)^k \leq 1$ for all $j \geq j_1$.
So, we have, $\forall\ j \geq j_1$,

$$\Omega_j \leq \frac{1}{j^{\frac{1}{k}}}.$$

Next, we show that $\{\zeta_n\}$ is a Cauchy sequence in G.
By triangular inequality, and $\forall\ m, n \geq n_0$

$$\mathsf{d}_\sigma(\zeta_n, \zeta_m)$$
$$\leq \sigma(\zeta_n, \zeta_{n+1}) \mathsf{d}_\sigma(\zeta_n, \zeta_{n+1}) + \sigma(\zeta_{n+1}, \zeta_m) \mathsf{d}_\sigma(\zeta_{n+1}, \zeta_m)$$

By repeatedly applying the triangle inequality, we get

$$\mathsf{d}_\sigma(\zeta_n, \zeta_m)$$
$$\leq \sigma(\zeta_n, \zeta_{n+1}) \mathsf{d}_\sigma(\zeta_n, \zeta_{n+1}) + \sigma(\zeta_{n+1}, \zeta_m) \sigma(\zeta_{n+1}, \zeta_{n+2}) \mathsf{d}_\sigma(\zeta_{n+1}, \zeta_{n+2})$$

$$+\sigma(\zeta_{n+1},\zeta_m)\sigma(\zeta_{n+2},\zeta_m)\sigma(\zeta_{n+2},\zeta_{n+3})\mathsf{d}_\sigma(\zeta_{n+2},\zeta_{n+3})+\sigma(\zeta_{n+1},\zeta_m)\sigma(\zeta_{n+2},\zeta_m)$$
$$\sigma(\zeta_{n+3},\zeta_m)\mathsf{d}_\sigma(\zeta_{n+3},\zeta_m)$$
$$\vdots$$
$$\leq \sigma(\zeta_n,\zeta_{n+1})\mathsf{d}_\sigma(\zeta_n,\zeta_{n+1})+\sum_{i=1+n}^{m-2}\left(\prod_{\mathrm{j}=1+n}^{\mathrm{i}}\sigma(\zeta_j,\zeta_m)\right)\sigma(\zeta_i,\zeta_{i+1})\mathsf{d}_\sigma(\zeta_i,\zeta_{i+1})+$$
$$\prod_{\mathrm{i}=1+n}^{\mathrm{m}-1}\sigma(\zeta_i,\zeta_m)\mathsf{d}_\sigma(\zeta_{m-1},\zeta_m)$$

$$\leq \sigma(\zeta_n,\zeta_{n+1})\mathsf{d}_\sigma(\zeta_n,\zeta_{n+1})+\sum_{i=1+n}^{m-2}\left(\prod_{\mathrm{j}=1+n}^{\mathrm{i}}\sigma(\zeta_j,\zeta_m)\right)\sigma(\zeta_i,\zeta_{i+1})\mathsf{d}_\sigma(\zeta_i,\zeta_{i+1})$$

$$+\left(\prod_{\mathrm{i}=1+n}^{\mathrm{m}-1}\sigma(\zeta_i,\zeta_m)\right)\sigma(\zeta_{m-1},\zeta_m)\mathsf{d}_\sigma(\zeta_{m-1},\zeta_m)$$

$$=\sigma(\zeta_n,\zeta_{n+1})\mathsf{d}_\sigma(\zeta_n,\zeta_{n+1})+\sum_{i=1+n}^{m-1}\left(\prod_{\mathrm{j}=1+n}^{\mathrm{i}}\sigma(\zeta_j,\zeta_m)\right)\sigma(\zeta_i,\zeta_{i+1})\mathsf{d}_\sigma(\zeta_i,\zeta_{i+1})$$
$$\leq \sigma(\zeta_n,\zeta_{n+1})\mathsf{d}_\sigma(\zeta_n,\zeta_{n+1})+\sum_{i=1+n}^{m-1}\left(\prod_{\mathrm{j}=0}^{\mathrm{i}}\sigma(\zeta_j,\zeta_m)\right)\sigma(\zeta_i,\zeta_{i+1})\mathsf{d}_\sigma(\zeta_i,\zeta_{i+1}).$$

Therefore,

$$\mathsf{d}_\sigma(\zeta_n,\zeta_m)$$
$$=\sigma(\zeta_n,\zeta_{n+1})\mathsf{d}_\sigma(\zeta_n,\zeta_{n+1})+\sum_{i=1+n}^{m-1}\left(\prod_{\mathrm{j}=0}^{\mathrm{i}}\sigma(\zeta_j,\zeta_m)\right)\sigma(\zeta_i,\zeta_{i+1})\frac{1}{i^{\frac{1}{k}}}. \tag{6.12}$$

Now

$$\sum_{i=1+n}^{m-1}\left(\prod_{\mathrm{j}=0}^{\mathrm{i}}\sigma(\zeta_j,\zeta_m)\right)\sigma(\zeta_i,\zeta_{i+1})\frac{1}{i^{\frac{1}{k}}}\leq \sum_{i=1+n}^{\infty}\frac{1}{i^{\frac{1}{k}}}\left(\prod_{\mathrm{j}=0}^{\mathrm{n}}\sigma(\zeta_j,\zeta_m)\right)\sigma(\zeta_i,\zeta_{i+1})$$
$$\leq \sum_{i=1+n}^{\infty}\mathsf{U}_i\mathsf{V}_i, \tag{6.13}$$

where $\mathsf{U}_i=\frac{1}{i^{\frac{1}{k}}}$ and $\mathsf{V}_i=\left(\prod_{\mathrm{j}=0}^{\mathrm{i}}\sigma(\zeta_j,\zeta_m)\right)\sigma(\zeta_i,\zeta_{i+1})$.

Since $\frac{1}{k}>0$, the series $\sum_{i=1+n}^{\infty}\frac{1}{i^{\frac{1}{k}}}$ converges. Also, $\{\mathsf{V}_i\}$ is an increasing sequence and bounded above, so $\lim_{i\to\infty}\{\mathsf{V}_i\}$ (which is nonzero) exists.

Consider the partial sum

$$\mathsf{R}_p = \sum_{i=1}^{p}\left(\prod_{\mathrm{j=0}}^{\mathrm{i}}\sigma(\zeta_j,\zeta_m)\right)\sigma(\zeta_i,\zeta_{i+1})\frac{1}{i^{\frac{1}{k}}},$$

Hence, for every $m > n$ we have

$$\mathsf{d}_\sigma(\zeta_n,\zeta_m) \leq [\sigma(\zeta_n,\zeta_{n+1})\mathsf{d}_\sigma(\zeta_n,\zeta_{n+1}) + \mathsf{R}_{\mathrm{m}-1} - \mathsf{R}_{\mathrm{n}}].$$

Now, in order to obtain the limit of R_p, we will find limit of the ratio $\frac{\mathsf{R}_{p+1}}{\mathsf{R}_p}$.

$$\begin{aligned}
\lim_{p\to\infty}\frac{\mathsf{R}_{p+1}}{\mathsf{R}_p} &= \lim_{p\to\infty}\frac{\sum_{i=0}^{p+1}\left(\prod_{j=0}^{i}\sigma(\zeta_j,\zeta_m)\right)\sigma(\zeta_i,\zeta_{i+1})}{\sum_{i=0}^{p}\left(\prod_{j=0}^{i}\sigma(\zeta_j,\zeta_m)\right)\sigma(\zeta_i,\zeta_{i+1})} \\
&= \lim_{p\to\infty}\frac{\prod_{j=0}^{p+1}\sigma(\zeta_j,\zeta_m)\sigma(\zeta_i,\zeta_{i+1})}{\sum_{i=0}^{p}\left(\prod_{j=0}^{i}\sigma(\zeta_j,\zeta_m)\right)\sigma(\zeta_i,\zeta_{i+1})} \\
&\leq \lim_{p\to\infty}\frac{\prod_{j=0}^{p+1}\sigma(\zeta_j,\zeta_m)\sigma(\zeta_i,\zeta_{i+1})}{\prod_{j=0}^{p}\sigma(\zeta_j,\zeta_m)\sigma(\zeta_i,\zeta_{i+1})}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{p\to\infty}\frac{\sigma(\zeta_{p+1},\zeta_m)\sigma(\zeta_{p+1},\zeta_{p+2})}{\sigma(\zeta_p,\zeta_{p+1})} \\
&< \frac{1}{\mathsf{P}} \quad \text{by using condition (6.1).}
\end{aligned} \tag{6.14}$$

So, by using (6.14) in (6.13) we obtain $\lim_{i\to\infty}\{\mathsf{U}_i\mathsf{V}_i\}$ exists and is finite. The convergence behavior of $\{\mathsf{U}_i\mathsf{V}_i\}$ is analogous to that demonstrated in Theorem 8 of [108].

Therefore by (6.14) we have

$$\Rightarrow \lim_{n,m\to\infty}\mathsf{d}_\sigma(\zeta_n,\zeta_m) = 0.$$

Thus, $\{\zeta_n\}$ is a Cauchy sequence in the orbitally complete controlled MS $(\mathsf{G}, \mathsf{d}_\sigma)$, so there is $\zeta^* \in \mathtt{O}(\zeta_0)$ such that $\{\zeta_n\} \longrightarrow \zeta^*$ as $n \to \infty$.
By (6.8) and $(\mathcal{F}_2)$,

$$\lim_{n\to\infty} \mathsf{d}_\sigma(\zeta_n, \Re_2(\zeta_n)) = 0.$$

Since $\zeta^* \mapsto \mathsf{d}_\sigma(\zeta_n, \Re_2(\zeta_n))$ is orbitally LSC and $\alpha(\zeta_n, \Re_2(\zeta_n)) > 1$. Therefore,

$$0 \leq \mathsf{d}_\sigma(\zeta^*, \Re_2(\zeta^*)) \leq \lim_{n\to\infty} \inf \mathsf{d}_\sigma(\zeta_n, \Re_2(\zeta_n)) \leq \lim_{n\to\infty} \inf \mathsf{d}_\sigma(\zeta_n, \zeta_{n+1}) = 0,$$

which leads to

$$\zeta^* \in \Re_2(\zeta^*).$$

Since, $\{\zeta_n\}$ is a Cauchy sequence in the orbitally complete controlled MS $(\mathsf{G}, \mathsf{d}_\sigma)$, so there is $\zeta^* \in \mathtt{O}(\zeta_0)$ such that $\{\zeta_n\} \longrightarrow \zeta^*$ as $n \to \infty$. By (6.8) and $(\mathcal{F}_2)$,

$$\lim_{n\to\infty} \mathsf{d}_\sigma(\zeta_n, \Re_1(\zeta_n)) = 0.$$

Since $\zeta^* \mapsto \mathsf{d}_\sigma(\zeta_n, \Re_1(\zeta_n))$ is orbitally LSC and $\alpha(\zeta_n, \Re_1(\zeta_n)) > 1$, one writes

$$0 \leq \mathsf{d}_\sigma(\zeta^*, \Re_1(\zeta^*)) \leq \lim_{n\to\infty} \inf \mathsf{d}_\sigma(\zeta_n, \Re_1(\zeta_n)) \leq \lim_{n\to\infty} \inf \mathsf{d}_\sigma(\zeta_n, \zeta_{n+1}) = 0,$$

which yields that

$$\zeta^* \in \Re_1(\zeta^*).$$

That is, $\Re_1$ and $\Re_2$ possess a common FP in G. □

Now, we prove the above result in an ordered controlled MS. If d_σ is a controlled metric on G and $(\mathsf{G}, \preccurlyeq)$ is a partially ordered set, then $(\mathsf{G}, \preccurlyeq, \mathsf{d}_\sigma)$ is called an ordered controlled MS. Also $\zeta_1, \zeta_2 \in \mathsf{G}$ are called comparable if either $\zeta_1 \preccurlyeq \zeta_2$ or $\zeta_2 \preccurlyeq \zeta_1$. The function $\Re_1 : \mathsf{G} \longrightarrow \mathrm{P}(\mathsf{G})$ is multivalued $\preccurlyeq$-dominated on G if $\zeta_1 \preccurlyeq \Re_1(\zeta_1)$ for any $\zeta_1 \in \mathsf{G}$.

Definition 6.2.3. Let $\Re_1, \Re_2 : \mathsf{G} \longrightarrow \mathrm{P}(\mathsf{G})$ be two MMs, $\mathtt{F} \in \nabla\mathtt{F}^*{}_\sigma$ and $\nu : (0, \infty) \longrightarrow (0, \infty)$. For all $\zeta_1, \zeta_2 \in \mathsf{G}$ with $\max\{\mathsf{d}_\sigma(\zeta_1, \Re_1(\zeta_1)), \mathsf{d}_\sigma(\zeta_2, \Re_2(\zeta_2))\} > 0$, define a set $\mathtt{F}^{\zeta_1}_{\nu,\preceq} \subset \mathsf{G}$ as

$$\mathtt{F}^{\zeta_1}_{\nu,\preceq} = \left\{ \begin{array}{c} \zeta_2 \in \Re_1(\zeta_1), \zeta_3 \in \Re_2(\zeta_2) : \mathtt{F}(\mathsf{d}_\sigma(\zeta_2,\zeta_3)) \\ \leq \mathtt{F}(\mathtt{M}(\zeta_1,\zeta_2)) + \mathcal{L}\Xi(\zeta_1,\zeta_2) + \nu(\mathsf{d}_\sigma(\zeta_2,\zeta_3)), \zeta_1 \preccurlyeq \zeta_2 \text{ and } \zeta_2 \preccurlyeq \zeta_3 \end{array} \right\},$$

where

$$\mathtt{M}(\zeta_1,\zeta_2) = \max\left\{ \mathsf{d}_\sigma(\zeta_1,\Re_1(\zeta_1)), \mathsf{d}_\sigma(\zeta_2,\Re_2(\zeta_2)), \frac{\{\mathsf{d}_\sigma(\zeta_1,\Re_1(\zeta_1)) \times \mathsf{d}_\sigma(\zeta_2,\Re_2(\zeta_2))\}}{1+\mathsf{d}_\sigma(\zeta_1,\zeta_2)} \right\},$$

and

$$\Xi(\zeta_1,\zeta_2) = \min\left\{ \mathsf{d}_\sigma(\zeta_1,\Re_1(\zeta_1)), \mathsf{d}_\sigma(\zeta_2,\Re_2(\zeta_2)), \mathsf{d}_\sigma(\zeta_1,\Re_2(\zeta_2)), \mathsf{d}_\sigma(\zeta_2,\Re_1(\zeta_1)) \right\}.$$

Theorem 6.2.4. Let $(\mathsf{G}, \preccurlyeq, \mathsf{d}_\sigma)$ be an ordered orbitally complete controlled MS. Let $\zeta_0 \in \mathsf{G}, \alpha : \mathsf{G} \times \mathsf{G} \longrightarrow [0,\infty)$ and $\Re_1, \Re_2 : \mathsf{G} \longrightarrow \mathrm{P}(\mathsf{G})$ be two α_*-dominated MMs and $\mathtt{F} \in \nabla\mathtt{F}^*{}_\sigma$. Suppose the following holds:

(i): $\displaystyle \sup_{m\geq 1} \lim_{i\to\infty} \frac{\sigma(\zeta_{i+1},\zeta_{i+2})\sigma(\zeta_{i+1},\zeta_m)}{\sigma(\zeta_i,\zeta_{i+1})} < \frac{1}{\mathtt{P}} \qquad \forall\, \mathtt{P} \in (0,1);$

(ii): the mapping $\zeta_3 \mapsto \max\{\mathsf{d}_\sigma(\zeta_1,\Re_1(\zeta_1)), \mathsf{d}_\sigma(\zeta_2,\Re_2(\zeta_2))\}$ is orbitally LSC;

(iii): $\exists$ functions $\tau, \nu : (0,\infty) \longrightarrow (0,\infty)$, such that

$$\tau(\mathsf{t}) > \nu(\mathsf{t}), \text{yields that } \lim_{\mathsf{s}\longrightarrow \mathsf{t}^+} \inf \tau(\mathsf{s}) > \lim_{\mathsf{s}\longrightarrow \mathsf{t}^+} \inf \nu(\mathsf{s})$$

$\forall\, \mathsf{t} \in (0,\infty)$;

(iv): for all $\zeta_1, \zeta_2 \in \{\Re_1\Re_2(\zeta_j)\}$ either $\zeta_1 \preccurlyeq \zeta_2$ or $\zeta_2 \preccurlyeq \zeta_1$ and $\max\{\mathsf{d}_\sigma(\zeta_1,\Re_1(\zeta_1)), \mathsf{d}_\sigma(\zeta_2,\Re_2(\zeta_2))\} > 0$, also $\{\Re_1\Re_2(\zeta_j)\} \longrightarrow \zeta^*$ there exist $\zeta_1, \zeta_2 \in \nabla\mathtt{F}^{\zeta_1}_{\nu,\preceq}$ satisfying;

$$\tau(\mathsf{d}_\sigma(\zeta_1,\zeta_2)) + \mathtt{F}(\mathtt{M}(\zeta_1,\zeta_2)) + \mathcal{L}\Xi(\zeta_1,\zeta_2) \leq \mathtt{F}(\mathsf{d}_\sigma(\zeta_1,\zeta_2)). \tag{6.15}$$

If (6.15) holds for ζ^*, $\zeta^* \preccurlyeq \zeta_j$ or $\zeta_j \preccurlyeq \zeta^*$ where $j = \{0,1,2,....\}$,

then $\Re_1$ and $\Re_2$ have a common FP.

Proof. Let $\alpha : \mathsf{G}\times\mathsf{G} \longrightarrow [0,\infty)$ be a function then for each $\tilde{\zeta} \in \mathsf{G}$, $\alpha(\tilde{\zeta},\zeta_1) = 1$ with $\tilde{\zeta} \preccurlyeq \zeta_1$, and for for each incomparable elements $\tilde{\zeta},\zeta_1 \in \mathsf{G}$ we have $\alpha(\tilde{\zeta},\zeta_1) = 0$. As $\Re_1$ and $\Re_2$ are two dominated MM on G, so $\tilde{\zeta} \preccurlyeq \Re_1(\zeta_1)$ and $\tilde{\zeta} \preccurlyeq \Re_2(\zeta_2)$ $\forall$ $\tilde{\zeta} \in \mathsf{G}$. This implies $\zeta \preccurlyeq \rho$ for every $\rho \in \Re_1(\zeta_1)$ and $\tilde{\zeta} \preccurlyeq \rho$ for each $\tilde{\zeta} \in \Re_2(\zeta_2)$. So, $\alpha(\tilde{\zeta},\rho) = 1$ for every $\rho \in \Re_1(\zeta_1)$ and $\alpha(\tilde{\zeta},\rho) = 1$ for every $\tilde{\zeta} \in \Re_2(\zeta_2)$.

$$\Rightarrow \inf\{\alpha(\tilde{\zeta},\rho) = 1 : \rho \in \Re_1(\zeta_1)\} = 1,$$

and

$$\inf\{\alpha(\tilde{\zeta},\rho) = 1 : \rho \in \Re_2(\zeta_2)\} = 1.$$

Hence

$$\alpha_*(\tilde{\zeta},\Re_1(\zeta_1)) = 1, \quad \alpha_*(\tilde{\zeta},\Re_2(\zeta_2)) = 1 \quad \text{for each } \tilde{\zeta} \in \mathsf{G}.$$

Since, $\Re_1,\Re_2$:$\mathsf{G} \longrightarrow \mathrm{P}(\mathsf{G})$ be two α_* dominated MM on G. Moreover, inequality (6.15) can be written as

$$\tau(\mathsf{d}_\sigma(\zeta_1,\zeta_2)) + \mathtt{F}(\mathtt{M}(\zeta_1,\zeta_2)) + \mathcal{L}\Xi(\zeta_1,\zeta_2) \leq \mathtt{F}(\mathsf{d}_\sigma(\zeta_1,\zeta_2)),$$

for each $\tilde{\zeta},\zeta_1 \in \{\Re_1\Re_2(\zeta_j)\}$, with either $\alpha(\tilde{\zeta},\zeta_1) \geq 1$ or $\alpha(\zeta_1,\tilde{\zeta}) \geq 1$. By Theorem 6.2.2, the sequence $\{\Re_1\Re_2(\zeta_j)\}$ is convergent in G that is $\{\Re_1\Re_2(\zeta_j)\} \longrightarrow \zeta^* \in \mathsf{G}$. Now, $\zeta_j,\zeta^* \in \mathsf{G}$ and either $\zeta_j \preccurlyeq \zeta^*$, or $\zeta^* \preccurlyeq \zeta_j$ implies that the either $\alpha_*(\zeta_j,\zeta^*) \geq 1$, or $\alpha_*(\zeta^*,\zeta_j) \geq 1$.

Hence, all the conditions of Theorem 6.2.2 are satisfied. So, both $\Re_1$ and $\Re_2$ have a FP $\zeta^* \in \mathsf{G}$. □

Example 6.2.5. Let $\mathsf{G} = [0,\infty)$, and $\preccurlyeq$ be a relation on G given by $\zeta_1 \preccurlyeq \zeta_2 \Leftrightarrow \zeta_1$ divides ζ_2. Then trivially $(\mathsf{G},\preccurlyeq)$ is partially ordered set. Define $\mathsf{d}_\sigma : \mathsf{G}\times\mathsf{G} \longrightarrow [0,\infty)$ by

$$\mathsf{d}_\sigma(\zeta_1,\zeta_2) = \begin{cases} 0, & \text{if } \zeta_1 = \zeta_2, \\ \frac{1}{\zeta_1}, & \text{if } \zeta_2 = 0, \\ 2(\frac{1}{\zeta_1} + \frac{1}{\zeta_2}), & \text{if } \zeta_1 \neq \zeta_2, \\ \frac{1}{\zeta_2}, & \text{if } \zeta_1 = 0. \end{cases}$$

Then $(\mathsf{d}_\sigma, \mathsf{G})$ is a complete controlled MS, where $\sigma : \mathsf{G}\times\mathsf{G} \longrightarrow (0,1]$ and $\alpha : \mathsf{G}\times\mathsf{G} \to \mathbb{R}$ are defined respectively by

$$\sigma(\zeta_1,\zeta_2) = \begin{cases} 1, & \text{if } \zeta_1,\zeta_2 \in [0,0.5), \\ 2+\zeta_1+\zeta_2, & \text{otherwise,} \end{cases}$$

and $\alpha(\zeta_1,\zeta_2) = \begin{cases} 1, & \text{if } \zeta_1 \le \zeta_2, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$

Also, take the mappings $\Re_1, \Re_2 : \mathsf{G} \longrightarrow \mathrm{P}(\mathsf{G})$ by

$$\Re_1(\zeta_1) = \{\zeta_1, 3\zeta_1, 5\zeta_1\},\ \Re_2(\zeta_2) = \{2\zeta_2, 4\zeta_2, 6\zeta_2\}.$$

Now, for all $\zeta_1,\zeta_2 \in \{\Re_1\Re_2(\zeta_j)\}$, with either $\alpha(\zeta_1,\zeta_2)\ge 1$, or $\alpha(\zeta_2,\zeta_1)\ge 1$.
$\mathsf{d}_\sigma(\zeta_1, \Re_1(\zeta_1)) = 0$ and
$\mathsf{d}_\sigma(\zeta_2, \Re_2(\zeta_2)) = \begin{cases} 0, & \text{if } \zeta_2 = 0, \\ \frac{7}{3\zeta_2}, & \text{if } \zeta_2 \ne 0. \end{cases}$

Therefore, $\zeta_2 \mapsto \max\{\mathsf{d}_\sigma(\zeta_1,\Re_1(\zeta_1)), \mathsf{d}_\sigma(\zeta_2,\Re_2(\zeta_2))\}$ is orbitally LSC. For $\zeta_2 \in \mathsf{G}$, we have $\zeta_1 = 3\zeta_2 \in \nabla\mathrm{F}^{\zeta_1}_{\nu,\preceq}$.
Then clearly $\mathsf{d}_\sigma(\zeta_1,\Re_2(\zeta_2)) = \frac{1}{\zeta_2}$ and $\mathsf{d}_\sigma(\zeta_2,\Re_1(\zeta_1)) = \frac{32}{15\zeta_2}$.
To verify

$$\tau(\mathsf{d}_\sigma(\zeta_1,\zeta_2)) + \mathtt{F}(\mathtt{M}(\zeta_1,\zeta_2)) + \mathcal{L}\Xi(\zeta_1,\zeta_2) \le \mathtt{F}(\mathsf{d}_\sigma(\zeta_1,\zeta_2)), \tag{6.16}$$

we need to consider,

$$\mathtt{M}(\zeta_1,\zeta_2) = \max\left\{\mathsf{d}_\sigma(\zeta_1,\Re_1(\zeta_1)), \mathsf{d}_\sigma(\zeta_2,\Re_2(\zeta_2)), \frac{\{\mathsf{d}_\sigma(\zeta_1,\Re_1(\zeta_1)) \times \mathsf{d}_\sigma(\zeta_2,\Re_2(\zeta_2))\}}{1+\mathsf{d}_\sigma(\zeta_1,\zeta_2)}\right\},$$

$$\begin{aligned}\mathtt{M}(\zeta_1,\zeta_2) &= \max\left\{0, \frac{7}{3\zeta_2}, 0\right\}, \\ &= \left\{\frac{7}{3\zeta_2}\right\},\end{aligned}$$

and

$$\Xi(\zeta_1,\zeta_2)=\min\left\{\mathsf{d}_\sigma(\zeta_1,\Re_1(\zeta_1)),\mathsf{d}_\sigma(\zeta_2,\Re_2(\zeta_2)),\mathsf{d}_\sigma(\zeta_1,\Re_2(\zeta_2)),\mathsf{d}_\sigma(\zeta_2,\Re_1(\zeta_1))\right\}$$

$$\begin{aligned}\Xi(\zeta_1,\zeta_2)=&\min\left\{0,\frac{7}{3\zeta_2},\frac{32}{15\zeta_2},\frac{1}{\zeta_2}\right\}\\ =&0.\end{aligned}$$

Choose $\mathtt{F}(t)=\ln(t)$, $\tau=\frac{1}{10}$ and $\nu=\frac{1}{15}$, $\tau>\nu$ and $\mathcal{L}=1$, (6.16) implies

$$\begin{aligned}&\frac{1}{10}+\frac{1}{15}+\ln\left(\frac{14}{3\zeta_2}\right)+0\\ &\leq\ln\left(\frac{64}{63}\frac{7}{3\zeta_2}\right)\\ &=\ln\left(\frac{448}{504}\frac{8}{3\zeta_2}\right)\\ &\leq\ln\left(\frac{8}{3\zeta_2}\right)\\ &=\mathtt{F}(\mathsf{d}_\sigma(\zeta_1,\zeta_2)).\end{aligned}\tag{6.17}$$

So all the axioms of Theorem 6.2.2 with $\mathcal{L}=1$ are satisfied.

6.3 Research Gap and Generalization

To highlight the advancement beyond existing results, we emphasize four key innovations in our work:

(1) : Example 6.1.4 depicts that the class of controlled MS is larger than that of b-MS. It is observed that results of Rasham et al. [62] are never generalized in controlled MS. In this research we fill that gap by generalizing those results to a larger class.

(2) : If we take $\sigma(\zeta_1,\zeta_3)=s$ in Definition 2.5.7, then controlled MS coincides with a b-MS. Consequently, the results of Rasham et al. [62] emerge as special cases of our broader framework.

(3) : If we take $\sigma(\zeta_1, \zeta_3) = s$, $\mathcal{L} = 0$, and $\mathtt{M}(\zeta_1, \zeta_2) = \mathsf{d}_\sigma(\zeta_1, \Re_1(\zeta_1))$ in Definition 2.5.7, our contraction condition coincides with the one of Nashine et al. [61] demonstrating that their theorems are subsumed by our more general approach.

(4) : If we take $\sigma(\zeta_1, \zeta_3) = s = 1$ in Definition 2.5.7, then controlled MS structure is simplified to an ordinary metric space. This reveals how our theorems not only generalize, but also refine numerous foundational results in standard metric FP theory.

6.4 Application to Integral Equations

Major findings of Rasham et al. [109] were used to investigate prerequisites for the existence of solution to a nonlinear integral equation. These results also demonstrate the existence of new FP for MM.

Let $\mathsf{G} = (\mathsf{C}[0,1], \mathbb{R}_+)$ be the set of continuous functions in $[0,1]$ endowed with the metric $\mathsf{d}_\sigma : \mathsf{G} \times \mathsf{G} \to [0, \infty)$ defined by

$$\mathsf{d}_\sigma(\mathtt{f}_1, \mathtt{f}_2) = \max_{\zeta \in [0,1]} |\mathtt{f}_1(\zeta) - \mathtt{f}_2(\zeta)|^5,$$

for all $\mathtt{f}_1, \mathtt{f}_2 \in \mathsf{C}[0,1]$. Then $(\mathsf{C}[0,1], \mathsf{d}_\sigma)$ is a complete controlled MS, where $\sigma : \mathsf{G} \times \mathsf{G} \longrightarrow (0,1]$ be defined by

$$\sigma(\zeta_1, \zeta_2) = \begin{cases} 1, & \text{if } \zeta_1, \zeta_2 \in [0, 0.5), \\ 2 + \zeta_1 + \zeta_2, & \text{otherwise,} \end{cases}$$

Define $\alpha : \mathsf{G} \times \mathsf{G} \to \mathbb{R}$ as

$$\alpha(\zeta_1, \zeta_2) = \begin{cases} 1, & \text{if } \zeta_1 \leq \zeta_2, \\ 0.5, & \text{otherwise.} \end{cases}$$

Consider the following integral equation:

$$\mathtt{f}(\zeta) = E(\zeta) + \int_0^{\zeta} \mathtt{K}(\zeta, \varrho, \mathtt{f}(\varrho))\, d\varrho \tag{6.18}$$

where $\mathtt{K} : [0,1] \times [0,1] \times \mathsf{G} \longrightarrow \mathbb{R}$ and $E : [0,1] \longrightarrow \mathbb{R}$ are continuous. We have to prove the existence of a solution of the equation (6.18) by applying Theorem 6.2.2.

Theorem 6.4.1. Let $\mathsf{G} = (\mathsf{C}[0,1], \mathbb{R})$ and $\Re_1, \Re_2 : \mathsf{G} \longrightarrow \mathsf{G}$ be the integral operators defined as follows:

$$\Re_1(\mathtt{f}_1(\zeta)) = E(\zeta) + \int_0^{\zeta} \mathtt{K}(\zeta, \varrho, \mathtt{f}_1(\varrho))\, d\varrho \quad \text{and}$$
$$\Re_2(\mathtt{f}_2(\zeta)) = E(\zeta) + \int_0^{\zeta} \mathtt{K}(\zeta, \varrho, \mathtt{f}_2(\varrho))\, d\varrho \;\; \textit{where } \mathtt{f}_1, \mathtt{f}_2 \in \mathsf{C}[0,1].$$

Assume the following hold:

(i): $\exists$ a continuous function $\mathfrak{u} : [0,1] \longrightarrow \mathbb{R}_+$ such that $\int_0^{\zeta} [\mathfrak{u}(\varrho)] d\varrho \;\leq e^{\frac{-\tau(t)}{5}}$ and for all $\mathtt{f}_1, \mathtt{f}_2 \in \mathsf{G}$ so that $\mathtt{f}_1(\zeta) \leq \mathtt{f}_2(\zeta)$ $\forall\, \zeta \in [0,1]$

$$|\mathtt{K}(\zeta, (\varrho), \mathtt{f}_1(\varrho)) - \mathtt{K}(\zeta, (\varrho), \mathtt{f}_2(\varrho))| \leq [\mathfrak{u}(\varrho)] \max_{\zeta \in [0,1]} |\mathtt{f}_1(\varrho) - \mathtt{f}_2(\varrho)|;$$

(ii): $\forall\, \mathtt{f}_1, \mathtt{f}_2, \mathtt{f}_3 \in \mathsf{G}$ such that $\mathtt{f}_1(\zeta) \leq \mathtt{f}_2(\zeta)$, we have $\mathtt{F}(\mathtt{M}(\mathtt{f}_1, \mathtt{f}_2)) + \mathcal{L}\Xi(\mathtt{f}_1, \mathtt{f}_2) + \nu(\mathsf{d}_\sigma(\mathtt{f}_2, \mathtt{z})) \leq |(\Re_1 \mathtt{f}_1)(\zeta) - (\Re_2 \mathtt{f}_2)(\zeta)|$, where

$$\mathtt{M}(\mathtt{f}_1, \mathtt{f}_2) = \max\left\{\mathsf{d}_\sigma(\mathtt{f}_1, \Re_1(\mathtt{f}_1)), \mathsf{d}_\sigma(\mathtt{f}_2, \Re_2(\mathtt{f}_2)), \frac{\{\mathsf{d}_\sigma(\mathtt{f}_1, \Re_1(\mathtt{f}_1)) \times \mathsf{d}_\sigma(\mathtt{f}_2, \Re_2(\mathtt{f}_2))\}}{1 + \mathsf{d}_\sigma(\mathtt{f}_1, \mathtt{f}_2)}\right\},$$

and

$$\Xi(\mathtt{f}_1, \mathtt{f}_2) = \min\left\{\mathsf{d}_\sigma(\mathtt{f}_1, \Re_1(\mathtt{f}_1)), \mathsf{d}_\sigma(\mathtt{f}_2, \Re_2(\mathtt{f}_2)), \mathsf{d}_\sigma(\mathtt{f}_1, \Re_2(\mathtt{f}_2)), \mathsf{d}_\sigma(\mathtt{f}_2, \Re_1(\mathtt{f}_1))\right\}.$$

Then integral equation (6.18) has a solution in $\mathsf{C}[0,1]$.

Proof. Let $\mathtt{f}_1, \mathtt{f}_2 \in \mathsf{G} = (\mathsf{C}[0,1], \mathbb{R}_+)$ and

$$\begin{aligned}
&|\Re_1(\mathtt{f}_1(\zeta)) - \Re_2(\mathtt{f}_2(\zeta))|^5 \\
=&\left| E(\zeta) + \int_0^\zeta \mathtt{K}(\zeta, \varrho, \mathtt{f}_1(\varrho))\, d\varrho \; - E(\zeta) + \int_0^\zeta \mathtt{K}(\zeta, \varrho, \mathtt{f}_2(\varrho))\, d\varrho \right|^5 \\
\leq&\left| \int_0^\zeta |\mathtt{K}_1(\zeta, (\varrho), \mathtt{f}_1(\varrho)) - \mathtt{K}_2(\zeta, (\varrho), \mathtt{f}_2(\varrho)) \right|^5 d\varrho \\
\leq& \int_0^\zeta [\mathtt{u}(\varrho)]^5 \max_{\zeta \in [0,1]} |\mathtt{f}_1(\varrho) - \mathtt{f}_2(\varrho)|^5 d\varrho \\
\leq& e^{-\tau(t)} \mathsf{d}_\sigma(\mathtt{f}_1, \mathtt{f}_2).
\end{aligned}$$

Thus, by (ii), we have

$$\mathtt{F}(\mathtt{M}(\mathtt{f}_1, \mathtt{f}_2)) + \mathcal{L}\Xi(\mathtt{f}_1, \mathtt{f}_2) + \nu(\mathsf{d}_\sigma(\mathtt{f}_2, \mathtt{f}_3)) \leq e^{-\tau(t)} \mathsf{d}_\sigma(\mathtt{f}_1, \mathtt{f}_2)$$

$$e^{\tau(t)} \left\{ \mathtt{F}(\mathtt{M}(\mathtt{f}_1, \mathtt{f}_2)) + \mathcal{L}\Xi(\mathtt{f}_1, \mathtt{f}_2) + \nu(\mathsf{d}_\sigma(\mathtt{f}_2, \mathtt{f}_3)) \right\} \leq \mathsf{d}_\sigma(\mathtt{f}_1, \mathtt{f}_2).$$

By taking ln on both sides,

$$\ln e^{\tau(t)} + \ln \left\{ \mathtt{F}(\mathtt{M}(\mathtt{f}_1, \mathtt{f}_2)) + \mathcal{L}\Xi(\mathtt{f}_1, \mathtt{f}_2) + \nu(\mathsf{d}_\sigma(\mathtt{f}_2, \mathtt{f}_3)) \right\} \leq \ln \mathsf{d}_\sigma(\mathtt{f}_1, \mathtt{f}_2).$$

That is,

$$\tau(t) + \ln \left\{ \mathtt{F}(\mathtt{M}(\mathtt{f}_1, \mathtt{f}_2)) + \mathcal{L}\Xi(\mathtt{f}_1, \mathtt{f}_2) + \nu(\mathsf{d}_\sigma(\mathtt{f}_2, \mathtt{f}_3)) \right\} \leq \ln \mathsf{d}_\sigma(\mathtt{f}_1, \mathtt{f}_2).$$

Define a function $\mathtt{F} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ by $\mathtt{F}(\mathsf{t}) = \ln(\mathsf{t}) \;\forall\; \mathsf{t} \in \mathbb{R}^+$ and for each $\mathsf{t} \in (0, \infty)$, $\tau(\mathsf{t}) = \frac{1}{10}$ and $\nu(\mathsf{t}) = \frac{1}{15}$.

Clearly, the mapping $\mathtt{F} \in \nabla\mathtt{F}^*{}_\sigma$ and

$$\tau(\mathsf{t}) > \nu(\mathsf{t}), \qquad \lim_{\mathsf{s} \longrightarrow \mathsf{t}^+} \inf \tau(\mathsf{s}) > \lim_{\mathsf{s} \longrightarrow \mathsf{t}^+} \inf \nu(\mathsf{s}).$$

Hence, the conditions of Theorem 6.2.2 are satisfied. That is, the operators $\Re_1$ and $\Re_2$ have a common FP. Thus, the integral equation (6.18) has a solution. □

Next, we give a numerical example by taking in particular $E(\zeta) = \frac{5}{6}\zeta$ and $\mathtt{K}(\zeta, \varrho, \mathtt{f}(\varrho)) = \zeta\mathtt{f}^5(\varrho)$. Thus, the integral equation (6.18) becomes

$$\mathtt{f}(\zeta) = \frac{5}{6}\zeta + \int_0^{\zeta} \zeta\mathtt{f}^5(\varrho)d\varrho. \tag{6.19}$$

Assume that for all $\mathtt{f}_1, \mathtt{f}_2 \in \mathsf{C}[0,1]$ and $\tau \in [1, +\infty)$, we have

$$|\zeta\mathtt{f}_1^5(\varrho) - \zeta\mathtt{f}_2^5(\varrho)| \leq e^{\frac{-\tau}{5}}|\mathtt{f}_1(\varrho) - \mathtt{f}_2(\varrho)|.$$

According to Theorem 6.2.2, the integral equation (6.19) admits a solution given by the function $\mathtt{f}(\zeta) = \zeta$. Figure 1 illustrates the convergence behavior of the iterative process, where the ζ values are plotted along the x-axis and the corresponding numerical approximations $\mathtt{f}_n(\zeta)$ are displayed on the y-axis.

$\mathtt{f}_0(\zeta) = 0$	$\mathtt{f}_1(\zeta) = \frac{5}{6}\zeta$
$\mathtt{f}_2(\zeta) = 0.900309\zeta$	$\mathtt{f}_3(\zeta) = 0.931914\zeta$
$\mathtt{f}_4(\zeta) = 0.950476\zeta$	$\mathtt{f}_5(\zeta) = 0.962617\zeta$
$\mathtt{f}_6(\zeta) = 0.971087\zeta$	$\mathtt{f}_7(\zeta) = 0.977256\zeta$
$\mathtt{f}_8(\zeta) = 0.981886\zeta$	$\mathtt{f}_9(\zeta) = 0.985439\zeta$
$\mathtt{f}_{10}(\zeta) = 0.98821\zeta$	$\mathtt{f}_{11}(\zeta) = 0.990401\zeta$
$\mathtt{f}_{12}(\zeta) = 0.992149\zeta$	$\mathtt{f}_{13}(\zeta) = 0.993556\zeta$
$\mathtt{f}_{14}(\zeta) = 0.994695\zeta$	$\mathtt{f}_{15}(\zeta) = 0.995623\zeta$
$\mathtt{f}_{16}(\zeta) = 0.996381\zeta$	$\mathtt{f}_{17}(\zeta) = 0.997002\zeta$
$\mathtt{f}_{18}(\zeta) = 0.997513\zeta$	$\mathtt{f}_{19}(\zeta) = 0.997935\zeta$
$\mathtt{f}_{20}(\zeta) = 0.998283\zeta$	$\mathtt{f}_{21}(\zeta) = 0.99857\zeta$
$\mathtt{f}_{22}(\zeta) = 0.998809\zeta$	$\mathtt{f}_{23}(\zeta) = 0.999006\zeta$
$\mathtt{f}_{24}(\zeta) = 0.99917\zeta$	$\mathtt{f}_{25}(\zeta) = 0.999306\zeta$
$\mathtt{f}_{26}(\zeta) = 0.999419\zeta$	$\mathtt{f}_{27}(\zeta) = 0.999513\zeta$

$\mathtt{f}_{28}(\zeta) = 0.999591\zeta$	$\mathtt{f}_{29}(\zeta) = 0.999656\zeta$
$\mathtt{f}_{30}(\zeta) = 0.99971\zeta$	$\mathtt{f}_{31}(\zeta) = 0.999755\zeta$
$\mathtt{f}_{32}(\zeta) = 0.999793\zeta$	$\mathtt{f}_{33}(\zeta) = 0.999824\zeta$
$\mathtt{f}_{34}(\zeta) = 0.99985\zeta$	$\mathtt{f}_{35}(\zeta) = 0.999871\zeta$
$\mathtt{f}_{36}(\zeta) = 0.99989\zeta$	$\mathtt{f}_{37}(\zeta) = 0.999905\zeta$
$\mathtt{f}_{38}(\zeta) = 0.999917\zeta$	$\mathtt{f}_{39}(\zeta) = 0.99993\zeta$
$\mathtt{f}_{40}(\zeta) = 0.999942\zeta$	$\mathtt{f}_{41}(\zeta) = 0.999948\zeta$
$\mathtt{f}_{42}(\zeta) = 0.999953\zeta$	$\mathtt{f}_{43}(\zeta) = 0.999958\zeta$
$\mathtt{f}_{44}(\zeta) = 0.999961\zeta$	$\mathtt{f}_{45}(\zeta) = 0.999964\zeta$

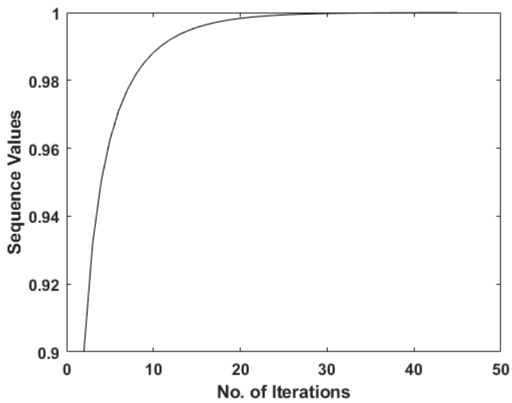

Figure 1: Convergence behavior of the numerical iterations

6.5 Application to Fractional Differential Equations

Fractional calculus is advancing rapidly, driven by the increasing importance of numerical solutions. This field remains a highly active area of research, with ongoing developments in numerical methods, for example the studies conducted by Rashid et al. [110] address the fluid dynamics models involving fractional operators. Pierre-Simon Lacroix laid the groundwork for fractional differentials in 1819, introducing a comprehensive framework that has since been built upon by numerous scholars. These researchers have successfully established various FP theorems in diverse metric spaces, leveraging generalized contractions and applying them to fractional differential equations (refer to [61, 107, 111] for notable examples). In this section, we investigate the existence of solution to one of such models within the context of controlled MS. Let $\Upsilon = [0,1]$ and $\mathsf{C}(\Upsilon, \mathbb{R})$ be the set of continuous functions defined on Υ. Define the metric $\mathsf{d}_\sigma : \mathsf{C}(\Upsilon, \mathbb{R}) \times \mathsf{C}(\Upsilon, \mathbb{R}) \to [0, \infty)$ by

$$\mathsf{d}_\sigma(\mathtt{f}_1, \mathtt{f}_2) = |\mathtt{f}_1 - \mathtt{f}_2|_\infty^5 = \max_{\zeta \in [0,1]} |\mathtt{f}_1(\zeta) - \mathtt{f}_2(\zeta)|^5,$$

for all $\mathtt{f}_1, \mathtt{f}_2 \in \mathsf{C}(\Upsilon, \mathbb{R})$. Then $(\mathsf{C}[0,1], \mathsf{d}_\sigma)$ is an orbitally complete controlled MS, where $\sigma : \mathsf{G} \times \mathsf{G} \longrightarrow (0,1]$ is defined by

$$\sigma(\zeta_1, \zeta_2) = \begin{cases} 1, & \text{if } \zeta_1, \zeta_2 \in [0, 0.5), \\ \zeta_1 + \zeta_2 + 2, & \text{otherwise.} \end{cases}$$

Define $\alpha : \mathsf{G} \times \mathsf{G} \to \mathbb{R}$ as

$$\alpha(\zeta_1, \zeta_2) = \begin{cases} 1, & \text{if } \zeta_1 \leq \zeta_2, \\ 0.5, & \text{otherwise.} \end{cases}$$

Let $\mathtt{K}_1, \mathtt{K}_2 : \Upsilon \times \mathbb{R} \longrightarrow \mathbb{R}$ be the two mappings such that

$$\mathtt{K}_1(\zeta, \mathtt{f}_1(\varrho)), \mathtt{K}_2(\zeta, \mathtt{f}_2(\varrho)) \geq 0 \text{ for all } \mathtt{f}_1, \mathtt{f}_2 \in \mathsf{C}(\Upsilon, \mathbb{R}) \text{ and } \zeta \in \Upsilon.$$

We will look into the following system of fractional differential equations:

$$^{\mathsf{C}}\mathrm{D}^{\mathtt{v}}\mathtt{f}_1(\zeta) = \mathtt{K}_1(\zeta, \mathtt{f}_1(\zeta)); \quad \mathtt{f}_1 \in \mathsf{C}(\Upsilon, \mathbb{R}) \tag{6.20}$$

$$^{\mathsf{C}}\mathrm{D}^{\mathtt{v}}\mathtt{f}_2(\zeta) = \mathtt{K}_2(\zeta, \mathtt{f}_2(\zeta)); \quad \mathtt{f}_2 \in \mathsf{C}(\Upsilon, \mathbb{R}), \tag{6.21}$$

with boundary conditions $\mathtt{f}_1(0) = 0, \Upsilon\mathtt{f}_1(1) = \mathtt{f}_1'(0), \mathtt{f}_2(0) = 0, \Upsilon\mathtt{f}_2(1) = \mathtt{f}_2'(0)$. Here, $^{\mathsf{C}}\mathrm{D}^{\mathtt{v}}$ represents the Caputo-Fabrizio derivative of order $\mathtt{v}$ given as

$$^{\mathsf{C}}\mathrm{D}^{\mathtt{v}}\mathtt{f}_1(\zeta) = \frac{1}{\Gamma(k-\mathtt{v})}\int_0^u (\zeta-\varrho)^{k-\mathtt{v}-1}\mathtt{f}_1(\varrho)d\varrho \quad (k-1 < \mathtt{v} < k, k = [n]+1),$$

and $\Upsilon^{\mathtt{v}}\mathtt{f}_1$ is defined by

$$\Upsilon^{\mathtt{v}}\mathtt{f}_1(\zeta) = \frac{1}{\Gamma(\mathtt{v})}\int_0^u (\zeta-\varrho)^{\mathtt{v}-1}\mathtt{f}_1(\varrho)d\varrho.$$

Then the equations (6.20) and (6.21) can be modified into

$$\mathtt{f}_1(\zeta) = \frac{1}{\Gamma(\mathtt{v})}\int_0^u (\zeta-\varrho)^{\mathtt{v}-1}\mathtt{K}_1(\varrho, \mathtt{f}_1(\varrho))d\varrho + \frac{\zeta}{\Gamma(\mathtt{v})}\int_0^{\mathsf{L}}\int_0^u (\varrho-\mathsf{j})^{\mathtt{v}-1}\mathtt{K}_1(\mathsf{j}, \mathtt{f}_1(\mathsf{j}))d\varrho d\mathsf{j},$$

and

$$\mathtt{f}_2(\zeta) = \frac{1}{\Gamma(\mathtt{v})}\int_0^u (\zeta-\varrho)^{\mathtt{v}-1}\mathtt{K}_2(\varrho, \mathtt{f}_2(\varrho))d\varrho + \frac{\zeta}{\Gamma(\mathtt{v})}\int_0^{\mathsf{L}}\int_0^u (\varrho-\mathsf{j})^{\mathtt{v}-1}\mathtt{K}_2(\mathsf{j}, \mathtt{f}_2(\mathsf{j}))d\varrho d\mathsf{j}.$$

Suppose that:

(i): there exists $\tau > 0$ such that

$$|\mathtt{K}_1(\zeta, \mathtt{f}_1(\varrho)) - \mathtt{K}_2(\zeta, \mathtt{f}_2(\varrho))| \leq \frac{e^{\frac{-\tau(t)}{5}}\Gamma(\mathtt{v}+1)}{5}|\mathtt{f}_1(\varrho) - \mathtt{f}_2(\varrho)|, \quad \forall\ \varrho \in \Upsilon;$$

(ii): $\mathsf{k}_0 \in \mathsf{C}(\Upsilon, \mathbb{R})$ so that for any $\zeta \in \Upsilon$,

$$\mathtt{f}_{1_0}(\zeta) \leq \frac{1}{\Gamma(\mathtt{v})}\int_0^u (\zeta-\varrho)^{\mathtt{v}-1}\mathtt{K}_1(\varrho, \mathtt{f}_{1_0}(\varrho))d\varrho + \frac{5\zeta}{\Gamma(\mathtt{v})}\int_0^{\mathsf{L}}\int_0^u (\varrho-\mathsf{j})^{\mathtt{v}-1}\mathtt{K}_1(\mathsf{j}, \mathtt{f}_{1_0}(\mathsf{j}))d\varrho d\mathsf{j}$$

and

$$\mathtt{f}_{2_0}(\zeta) \leq \frac{1}{\Gamma(\mathtt{v})}\int_0^u (\zeta-\varrho)^{\mathtt{v}-1}\mathtt{K}_2(\varrho, \mathtt{f}_{2_0}(\varrho))d\varrho + \frac{5\zeta}{\Gamma(\mathtt{v})}\int_0^{\mathsf{L}}\int_0^u (\varrho-\mathsf{j})^{\mathtt{v}-1}\mathtt{K}_2(\mathsf{j}, \mathtt{f}_{2_0}(\mathsf{j}))d\varrho d\mathsf{j};$$

(iii): Let $\mathsf{G} = \{\mathtt{f} \in \mathsf{C}(\Upsilon, \mathbb{R}) : \mathtt{f}(\zeta) \geq 0\}$ for all $\zeta \in \Upsilon$, and define the operator $\Re_1, \Re_2 : \mathsf{G} \longrightarrow \mathsf{G}$ by

$$(\Re_1\mathtt{f}_1)(\zeta) = \frac{1}{\Gamma(\mathtt{v})}\int_0^u (\zeta-\varrho)^{\mathtt{v}-1}\mathtt{K}_1(\varrho, \mathtt{f}_1(\varrho))d\varrho + \frac{5\zeta}{\Gamma(\mathtt{v})}\int_0^{\mathsf{L}}\int_0^u (\varrho-\mathsf{j})^{\mathtt{v}-1}\mathtt{K}_1(\mathsf{j}, \mathtt{f}_1(\mathsf{j}))d\varrho d\mathsf{j}$$

and

$$(\Re_2\mathtt{f}_2)(\zeta) = \frac{1}{\Gamma(\mathtt{v})}\int_0^u (\zeta-\varrho)^{\mathtt{v}-1}\mathtt{K}_2(\varrho, \mathtt{f}_2(\varrho))d\varrho + \frac{5\zeta}{\Gamma(\mathtt{v})}\int_0^{\mathsf{L}}\int_0^u (\varrho-\mathsf{j})^{\mathtt{v}-1}\mathtt{K}_2(\mathsf{j}, \mathtt{f}_2(\mathsf{j}))d\varrho d\mathsf{j},$$

satisfying:

$$\mathtt{F}(\mathtt{M}(\mathtt{f}_1, \mathtt{f}_2)) + \mathcal{L}\Xi(\mathtt{f}_1, \mathtt{f}_2) + \nu(\mathsf{d}_\sigma(\mathtt{f}_2, \mathtt{f}_3)) \leq |(\Re_1\mathtt{f}_1)(\zeta) - (\Re_2\mathtt{f}_2)(\zeta)|^5,$$

where

$$\mathtt{M}(\mathtt{f}_1, \mathtt{f}_2) = \max\left\{\mathsf{d}_\sigma(\mathtt{f}_1, \Re_1(\mathtt{f}_1)), \mathsf{d}_\sigma(\mathtt{f}_2, \Re_2(\mathtt{f}_2)), \frac{\{\mathsf{d}_\sigma(\mathtt{f}_1, \Re_1(\mathtt{f}_1)) \times \mathsf{d}_\sigma(\mathtt{f}_2, \Re_2(\mathtt{f}_2))\}}{1 + \mathsf{d}_\sigma(\mathtt{f}_1, \mathtt{f}_2)}\right\}$$

and

$$\Xi(\mathtt{f}_1, \mathtt{f}_2) = \min\left\{\mathsf{d}_\sigma(\mathtt{f}_1, \Re_1(\mathtt{f}_1)), \mathsf{d}_\sigma(\mathtt{f}_2, \Re_2(\mathtt{f}_2)), \mathsf{d}_\sigma(\mathtt{f}_1, \Re_2(\mathtt{f}_2)), \mathsf{d}_\sigma(\mathtt{f}_2, \Re_1(\mathtt{f}_1))\right\}.$$

Theorem 6.5.1. If the conditions $(i) - (iii)$ are fulfilled, then a common solution in $\mathsf{C}(\Upsilon, \mathbb{R})$ for equations (6.20) and (6.21) exists.

Proof. Consider,

$$|(\Re_1 \mathtt{f}_1)(\zeta) - (\Re_2 \mathtt{f}_2)(\zeta)| =$$
$$|\frac{1}{\Gamma(\mathtt{v})}\int_0^u (\zeta-\varrho)^{\mathtt{v}-1}\mathtt{K}_1(\varrho,\mathtt{f}_1(\varrho))d\varrho - \frac{1}{\Gamma(\mathtt{v})}\int_0^u (\zeta-\varrho)^{\mathtt{v}-1}\mathtt{K}_2(\varrho,\mathtt{f}_2(\varrho))d\varrho +$$

$$\frac{5\zeta}{\Gamma(\mathtt{v})}\int_0^{\mathsf{L}}\int_0^{\mathsf{z}} (\mathsf{z}-\mathsf{j})^{\mathtt{v}-1}\mathtt{K}_1(\mathsf{j},\mathtt{f}_1(\mathsf{j}))d\mathsf{z}d\mathsf{j} - \frac{5\zeta}{\Gamma(\mathtt{v})}\int_0^{\mathsf{L}}\int_0^{\mathsf{z}} (\mathsf{z}-\mathsf{j})^{\mathtt{v}-1}\mathtt{K}_2(\mathsf{j},\mathtt{f}_2(\mathsf{j}))d\mathsf{z}d\mathsf{j}|.$$

One writes

$$\begin{aligned}
&|(\Re_1 \mathtt{f}_1)(\zeta) - (\Re_2 \mathtt{f}_2)(\zeta)| \\
&\leq |\int_0^u \{\frac{1}{\Gamma(\mathtt{v})}(\zeta-\varrho)^{\mathtt{v}-1}\mathtt{K}_1(\varrho,\mathtt{f}_1(\varrho)) - \frac{1}{\Gamma(\mathtt{v})}(\zeta-\varrho)^{\mathtt{v}-1}\mathtt{K}_2(\varrho,\mathtt{f}_2(\varrho))\}d\varrho| \\
&+ |\int_0^{\mathsf{L}}\int_0^{\mathsf{z}} \{\frac{5\zeta}{\Gamma(\mathtt{v})}(\mathsf{z}-\mathsf{j})^{\mathtt{v}-1}\mathtt{K}_1(\mathsf{j},\mathtt{f}_1(\mathsf{j})) - \frac{5\zeta}{\Gamma(\mathtt{v})}(\mathsf{z}-\mathsf{j})^{\mathtt{v}-1}\mathtt{K}_2(\mathsf{j},\mathtt{f}_2(\mathsf{j}))\}d\mathsf{z}d\mathsf{j}| \\
&\leq \frac{1}{\Gamma(\mathtt{v})}\frac{e^{\frac{-\tau(t)}{5}}\Gamma(\mathtt{v}+1)}{5}\int_0^u (\zeta-\varrho)^{\mathtt{v}-1}|\mathtt{f}_1(\varrho)) - \mathtt{f}_2(\varrho))|d\varrho \\
&+ \frac{5}{\Gamma(\mathtt{v})}\frac{e^{\frac{-\tau(t)}{5}}\Gamma(\mathtt{v}+1)}{5}\int_0^{\mathsf{L}}\int_0^{\mathsf{z}} \frac{2\zeta}{\Gamma(\mathtt{v})}(\mathsf{z}-\mathsf{j})^{\mathtt{v}-1}|\mathtt{f}_1(\mathsf{j})) - \mathtt{f}_2(\mathsf{j}))|d\mathsf{z}d\mathsf{j} \\
&\leq \frac{1}{\Gamma(\mathtt{v})}\frac{e^{\frac{-\tau(t)}{5}}\Gamma(\mathtt{v}+1)}{5}|\mathtt{f}_1(\varrho)) - \mathtt{f}_2(\varrho))|\int_0^u (\zeta-\varrho)^{\mathtt{v}-1}d\varrho \\
&+ \frac{5}{\Gamma(\mathtt{v})}\frac{e^{\frac{-\tau(t)}{5}}\Gamma(\mathtt{v}+1)\Gamma(\mathtt{v})}{5\Gamma(\mathtt{v})\Gamma(\mathtt{v}+1)}|\mathtt{f}_1(\mathsf{j})) - \mathtt{f}_2(\mathsf{j}))| \\
&\int_0^{\mathsf{L}}\int_0^{\mathsf{z}} \frac{2\zeta}{\Gamma(\mathtt{v})}(\mathsf{z}-\mathsf{j})^{\mathtt{v}-1}|\mathtt{f}_1(\mathsf{j})) - \mathtt{f}_2(\mathsf{j}))|d\mathsf{z}d\mathsf{j} \\
&\leq \frac{e^{\frac{-\tau(t)}{5}}\Gamma(\mathtt{v}+1)\Gamma(\mathtt{v})}{\Gamma(\mathtt{v})\Gamma(\mathtt{v}+1)}|\mathtt{f}_1(\varrho)) - \mathtt{f}_2(\varrho))| \\
&\quad + 2e^{-\tau(t)}\mathcal{B}(\mathtt{v}+1,1)\frac{\Gamma(\mathtt{v}+1)\Gamma(\mathtt{v})}{\Gamma(\mathtt{v})\Gamma(\mathtt{v}+1)}|\mathtt{f}_1(\varrho)) - \mathtt{f}_2(\varrho))| \\
&\leq e^{\frac{-\tau(t)}{5}}|\mathtt{f}_1(\varrho)) - \mathtt{f}_2(\varrho))| + e^{\frac{-\tau(t)}{5}}|\mathtt{f}_1(\varrho)) - \mathtt{f}_2(\varrho))| \leq e^{\frac{-\tau(t)}{5}}|\mathtt{f}_1(\varrho)) - \mathtt{f}_2(\varrho))|,
\end{aligned}$$

where $\mathcal{B}$ is beta function. This further implies that

$$|(\Re_1 \mathtt{f}_1)(\zeta) - (\Re_2 \mathtt{f}_2)(\zeta)| \leq e^{\frac{-\tau(t)}{5}}|\mathtt{f}_1(\varrho)) - \mathtt{f}_2(\varrho))|.$$

By taking power 5 on both sides, we get

$$|(\Re_1 \mathtt{f}_1)(\zeta) - (\Re_2 \mathtt{f}_2)(\zeta)|^5 \leq e^{-\tau(t)}|\mathtt{f}_1(\varrho)) - \mathtt{f}_2(\varrho))|^5.$$

By assumption (iii), the above inequality written as

$$\mathtt{F}(\mathtt{M}(\mathtt{f}_1,\mathtt{f}_2)) + \mathcal{L}\Xi(\mathtt{f}_1,\mathtt{f}_2) + \nu(\mathtt{d}_\sigma(\mathtt{f}_2,\mathtt{f}_3)) \leq e^{-\tau(t)}|(\Re_1\mathtt{f}_1)(\zeta) - (\Re_2\mathtt{f}_2)(\zeta)|^5, \quad (6.22)$$

where

$$\mathtt{M}(\mathtt{f}_1,\mathtt{f}_2) = \max\left\{\mathtt{d}_\sigma(\mathtt{f}_1,\Re_1(\mathtt{f}_1)), \mathtt{d}_\sigma(\mathtt{f}_2,\Re_2(\mathtt{f}_2)), \frac{\{\mathtt{d}_\sigma(\mathtt{f}_1,\Re_1(\mathtt{f}_1)) \times \mathtt{d}_\sigma(\mathtt{f}_2,\Re_2(\mathtt{f}_2))\}}{1+\mathtt{d}_\sigma(\mathtt{f}_1,\mathtt{f}_2)}\right\}$$

and

$$\Xi(\mathtt{f}_1,\mathtt{f}_2) = \min\left\{\mathtt{d}_\sigma(\mathtt{f}_1,\Re_1(\mathtt{f}_1)), \mathtt{d}_\sigma(\mathtt{f}_2,\Re_2(\mathtt{f}_2)), \mathtt{d}_\sigma(\mathtt{f}_1,\Re_2(\mathtt{f}_2)), \mathtt{d}_\sigma(\mathtt{f}_2,\Re_1(\mathtt{f}_1))\right\}.$$

Define a function $\mathtt{F} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ by $\mathtt{F}(\mathtt{t}) = \ln(\mathtt{t})\ \forall\ \mathtt{t} \in \mathbb{R}^+$ and $\tau(\mathtt{t}) = \frac{1}{10}$ and $\nu(\mathtt{t}) = \frac{1}{15}$ for each $\mathtt{t} \in \mathbb{R}^+$. So the mapping $\mathtt{F} \in \nabla\mathtt{F}^*{}_\sigma$ and

$$\tau(\mathtt{t}) > \nu(\mathtt{t}), \qquad \lim_{\mathtt{t}\longrightarrow\mathtt{t}^+} \inf \tau(\varrho) > \lim_{\mathtt{t}\longrightarrow\mathtt{t}^+} \inf \nu(\varrho).$$

So, (6.22) becomes

$$\ln e^\tau + \ln\left\{\mathtt{F}(\mathtt{M}(\mathtt{f}_1,\mathtt{f}_2)) + \mathcal{L}\Xi(\mathtt{f}_1,\mathtt{f}_2) + \nu(\mathtt{d}_\sigma(\mathtt{f}_2,\mathtt{f}_3))\right\} \leq \ln \mathtt{d}_\sigma(\mathtt{f}_1,\mathtt{f}_2).$$

That is,

$$\tau + \ln\left\{\mathtt{F}(\mathtt{M}(\mathtt{f}_1,\mathtt{f}_2)) + \mathcal{L}\Xi(\mathtt{f}_1,\mathtt{f}_2) + \nu(\mathtt{d}_\sigma(\mathtt{f}_2,\mathtt{f}_3))\right\} \leq \ln \mathtt{d}_\sigma(\mathtt{f}_1,\mathtt{f}_2).$$

Hence, all the conditions of Theorem 6.2.2 are satisfied. Hence, the equations (6.20) and (6.21) admit a common solution in $\mathtt{C}(\Upsilon,\mathbb{R})$. □

6.6 Conclusion

This research pioneers new FP results. These results generalizes many existing results in literature as mentioned in section 6.3. In the future, the generalization of our work can be done by considering different abstract spaces like double controlled MSs and triple controlled MSs. One can also modify the extended rational type advanced Nashine-Wardowski-Feng-Liu-type contraction condition to obtain more general results. Given the growing applications of fractional differential equations in modeling real world phenomena, future research could develop solution methods using FP techniques.

Chapter 7

Conclusion

In this dissertation, we have developed and unified several novel FP results for fuzzy and MMs within various generalized metric frameworks, including b-MSs, double controlled MSs, and orbitally complete controlled MSs. Motivated by the need to generalize classical contraction principles, we introduced new contractive conditions such as the (P, ψ)-type contraction for FMs in complete b-MSs, along with Θ-fuzzy double controlled and almost generalized double controlled contractions. The results established in each setting not only extend but also encompass a wide array of classical FP theorems. Several non-trivial examples have been constructed to illustrate the validity and applicability of the main theorems.

In addition to theoretical development, we applied the proposed FP results to boundary value problems and stochastic Volterra integral equations. These applications highlight the practical significance of FP theory in solving nonlinear and integral equations, thereby bridging the gap between abstract mathematical theory and real-world problems.

Furthermore, we formulated common FP theorems for pairs of fuzzy and MMs using advanced integral-type and rational-type contractions. These results were developed in the framework of orbitally complete controlled MSs, offering a versatile foundation for future research involving ordered or structured MSs.

Overall, this dissertation contributes a comprehensive and generalized framework for FP theory, which not only advances the existing body of knowledge but also opens new directions for research and application in nonlinear analysis, differential equations, and dynamical systems.

7.1 Future Work

In the future, one can extend our work in two directions:

(i): By generalizing the contraction condition, for example, by introducing new types such as tower-type contractions.

(ii): By broadening the scope of the underlying space, particularly by considering controlled and double controlled MSs.

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