THEORY OF DYNAMIC INTEGRAL SLIDING MODE AND ITS APPLICATIONS

by

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Dedicated to Zainab Khan

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ABSTRACT

Sliding Mode Control(SMC) theory, being a robust control technique, has a variety of applications in industry. It showed fruitful results since its introduction. However, the control law suffers from the well known chattering phenomena which may cause problems in applications. Despite the robust nature of SMC, it becomes sensitive to uncertainties in reaching phase in certain applications. This sensitivity may result in marginal stability or even in the instability of the system. Thus the reaching phase elimination may enhance the robustness. In addition, in some real applications, it is needed to have the settling time as small as possible. Therefore, efforts were devoted to solve these main issues. The existing literature of sliding mode control is rich in methods used for chattering attenuation robustness enhancement and performance improvement. They solved some of the discussed issues at the cost of the other and vice versa. For example, the chattering reduction resulted in robustness loss as well as performance loss and vice versa.

In this thesis, it is tried to have robust performance with reduced chattering. For this purpose, a novel output feedback based sliding mode strategy is proposed for uncertain nonlinear systems which is based on the existing theory of dynamic sliding mode and basic theory of integral sliding mode control. The proposed cont enhances the performance of the system while rejecting the uncertainties.

The proposed control law is designed for both SISO and MIMO uncertain nonlinear systems. The claims of the proposed technique are verified via some examples. Furthermore, this control algorithm is extended to both SISO and MIMO nonlinear systems operating under matched and unmatched state dependent uncertainties.

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DISMC	Dynamic Integral Sliding Mode Control
DSMC	Dynamic Sliding Mode Control
HOSMC	Higher Order Sliding Mode Control
ISMC	Integral Sliding Mode Control
LRD	Levant Robust Differentiator
NLS	Nonlinear System
SHGO	Semi High Gain Observer
SMC	Sliding Mode Control
TSMC	Terminal Sliding Mode Control
TTS	Three Tank System
LGCC	Locally Generalized Controllable Canonical

Chapter 1

INTRODUCTION

Modern world has a wide class of dynamic systems which are operated either manually or automatically. The automatic handling of these systems made the life of mankind very comfortable. The automatic functioning of these systems are based on some input signals called the controlled signals. These control signals are produced by using some control design strategies/algorithms. The design of the algorithms varies according to the circumstances and the requirements of the system. The field of control theory has wide variety of these methodologies. One of these schemes is Sliding Mode Control (SMC) technique which is famous for its robust nature and easy design for both linear and nonlinear systems. This chapter briefly introduces SMC, motivations of the research work and contributions of the thesis followed by the overview of forthcoming chapters.

The tradition of Control theory research is very long and distinguished which stretches back to nineteenth-century dynamics and stability theory. It arised as an engineering discipline in late 1950s. Nyquist, Bode and Wiener were the pioneers, among the others, who worked on frequency analysis based controller design. In the middle of nineteenth-century, Emelyanov and his co-worker felt that the conventional state feedback techniques were not robust against disturbances and nonlinearities. They developed a variable structure control (VSC); the so-called sliding mode control. The theory of SMC plays a vital role in variable structure system (VSS) theory. It emerged as a technique capable of use in given uncertain control systems ([1], [2]).

The basic idea of this technique is to enforce sliding mode in the system's state space. The sliding mode occurs along a constraint the so-called sliding manifold. These sliding manifolds are normally constructed as a surface or intersection of surfaces in the state space which are also termed as switching surface. In sliding mode robustness is guaranteed against certain class of uncertainties, un-modeled dynamics, parametric uncertainties and external disturbances ([3], [4], [5]). However, it experiences chattering phenomena, associated with high frequency vibrations across the sliding manifold, which leads to the damage in actuators and system itself. The recent research strategies are widely addressed in [6]. All of these variants of the sliding mode control have some common steps in the design procedure. In the first step a switching manifold is designed which may also be called an invariant set or intersection of invariant sets. In the second step, a switching control law is designed which keeps the system states on the sliding manifold. Traditionally, SMC occurs in two phases which are termed as 'reaching phase' and 'sliding phase'. As the name indicates, in reaching phase, all the system's states are converged to the sliding manifold. On the other hand, in sliding phase, these states are kept over the sliding manifold until the states reach the equilibrium position. In sliding mode the system operates with reduced order dynamics which play a significant role in the robustness of the control law. Once the system states reach the sliding manifold they become insensitive to disturbance and uncertainties. Sliding mode proved to be a robust control technique and it showed appealing results in the real applications, but the chattering across the switching manifold was a challenging problem which occurs when sliding mode is established. A large number of methods were proposed to overcome the aforementioned problems. The other issues, which were focused, include the trade off between chattering and robustness as well as accuracy. In the present age the researchers are strongly associated to robustness enhancement, chattering alleviation and performance. In the forthcoming, section we will discuss the motivation of the work presented in this monograph.

1.1 Motivations of the Work

The control of uncertain systems via SMC techniques had been the main focus of researcher for the last three decades. Since the introduction of sliding mode, sliding mode gave birth to a wide number of variants. The first challenging job, as already mentioned, for the researchers was the chattering attenuation. This objective was achieved either with the use of some low pass filters or some other techniques like higher order sliding mode control (HOSMC) (see for instance, [5], [7], [8] and [9]) and sliding sector method (see for detail, [10], [11] and [12] and the reference therein). But the main issue was the loss of robustness which arose while attenuating the effects of chattering phenomena. The approach of HOSMC exactly preserve the key features of SMC and provide chattering free control input signal (or results in substantially reduced chattering) . This technique brought revolution in theory of sliding modes but the key issue which appeared was the loss of robustness [5]. However, robustness was provided with the use of some robust differentiators [13]. This helps in robust and accurate measurement of output and its derivatives. Among all the chattering attenuation efforts, Dynamic Sliding Mode Control (DSMC)(see for instance, [14], [15] and [16]) was attempted to provide chattering free robust control law. This provided satisfactory results for a class of nonlinear systems with enhanced robustness and attenuated chattering. The above discussed control design methodologies mainly focused on the robustness and chattering attenuation problems.

Majority of the nonlinear systems show very sensitive behavior to very small disturbances, even of matched nature, in the reaching phase. This sensitivity of the system in reaching phase of SMC strategy may result in undesirable results or even in the instability of the system. Thus it was needed to have a reaching phase free sliding mode. The first attempt, which is named as Integral Sliding Mode Control

(ISMC), was made in [17] to eliminate reaching phase. The existing literature contains countable efforts which handle simultaneously, the robustness enhancement, performance improvement and chattering reduction. Each of the attempts have some merits along with some demerits.

Therefore, the aforementioned discussions motivates to propose a control design strategy which can provide us robust control with alleviated chattering and performance. The approach should be able to meet the above three issues in optimal sense. The optimality, in this paragraph, is meant to have acceptable performance with considerably reduced chattering and improved robustness at a time. In the following section the requirements are outlined. In order to meet the above discussed requirements, a control design algorithm development is required. The development will be made if all of the following condition fulfills.

- The output must be measurable
- The output derivatives must be observable
- The uncertainties must be norm bounded
- The continuous control component (this will appear in the control design) must have good performance
- The gains of the discontinuous control component must be selected according to the uncertainties bounds

If any of the above conditions is not satisfied, one may not be able to have the desired performance with tolerable chattering phenomenon and acceptable robustness. The output measurement provision is of prime importance for output feedback techniques. The good estimation of the output results provides very good control of the output of the system. If either the output or its derivative results in a non accurate estimate, the control objective seems much difficult to meet.

The uncertainties must be norm bounded otherwise, the instability may be the net result. The fast convergence to the desired value of the output is based on the performance of the continuous control component. Therefore, it must be designed in a rigorous way. The last requirement is that the discontinuous control component gain must be sufficiently greater than the bounds of the uncertainties to reject the unwanted effects of the uncertainties. These gain may be selected according to some conditions which will result in a satisfactory control of the output. In the following section the main contributions, in the form of control law development (for performance improvement, chattering attenuation with robustness enhancement), are listed.

1.2 Statement of Contributions

The major contributions, of the research work presented in this manuscript, is the development of a control algorithm for nonlinear systems. This proposed scheme is based on the DSMC and ISMC synthesis. This inherits the good features of both the techniques. The chattering attenuation occurs due to DSMC with partial contribution from ISMC, robustness depends on both DSMC and ISMC and performance comes from the continuous control component which is required in ISMC. Consequently, the proposed control design method which is named as Dynamic Integral Sliding Mode Control (DISMC), asymptotically regulates the output to origin in the presence of uncertainties. The unwanted vibrations/chattering against the sliding manifold is considerably alleviated with improved performance and enhanced robustness. The uncertainties appearing in the nonlinear systems may be caused by the unmodeled dynamic, external disturbance and parametric variations. The nature of these uncertainties may either be matched (uncertainties which disturb the system through the same channel where the control input is applied to the system) or unmatched (uncertainties which acts upon the system at the nodes other than the control input node).

Following are the main contributions of the present manuscript.

- Control of a SISO Uncertain Nonlinear Dynamic System via Dynamic Integral Sliding Mode Control [18]
- DISMC Control of SISO Nonlinear System Operating under state dependent Unmatched Uncertainties ([19], [20])
- Control of a MIMO Uncertain Nonlinear Dynamic System via Dynamic Integral Sliding Mode Control [21]
- DISMC Control of MIMO Nonlinear System Operating under Unmatched Uncertainties [22]

1.3 Disseminations

The outcome of the PhD research work, published in journals and proceeding of international conferences, is listed below.

Journal Publications

1. Qudrat Khan, Aamer Iqbal Bhatti, Mohammad Iqbal, Qadeer Ahmed, ”Dynamic Integral Sliding Mode Control of SISO Uncertain Nonlinear Systems”, International Journal of Innovative Computing, Information and Control, Vol.8, Issue 7, July 2012, ISSN 1349-4198.
2. Qudrat Khan, Aamer Iqbal Bhatti,Sohail Iqbal, Mohammad Iqbal, ”Dynamic Integral Sliding Mode Control of MIMO Uncertain Nonlinear Systems”, Int. Journal of Control Automation and Systems, 44(1):1105 1120, feb 2011.

3. Qudrat Khan, Aamer Iqbal Bhatti and Antonella Ferrara, "Integral manifold based design of a dynamic sliding mode controller for SISO nonlinear uncertain systems" Under the Review of International Journal of Control.

4. Qudrat Khan, Aamer Iqbal Bhatti and Antonella Ferrara, "Integral manifold based design of a dynamic sliding mode controller for MIMO nonlinear uncertain systems" Under the Review of IET, Control Theory and Applications.

5. Mohammad Iqbal, Aamer Iqbal Bhatti, Sohail Iqbal, Qudrat Khan, "Robust Parameter Estimation of Nonlinear Systems using Sliding Mode Differentiator Observer" IEEE Transaction on Industrial Electronics, 58(2), pp. 680 - 689, ISI Thomson IF: 5.468

6. Q. Ahmed, A.I. Bhatti, Q. Khan and M. Raza. Condition Monitoring of Gasoline Engine Air Intake system using Second Order Sliding Modes. Vehicle Design, International Journal of 2012, impact factor, 0.57

Conference Publications

1. Qudrat Khan, A.I. Bhatti and Qadeer Ahmed, "Dynamic Integral Sliding Mode Control of Nonlinear Systems with Mismatched Uncertainties and Time Varying Disturbances", IFAC World Congress Milano (Italy) August 28 - September 2, 2011.

2. M. Iqbal, A.I.Bhatti, Q.Khan " Dynamic Sliding Mode Control for Uncertain Three Tank System" INMIC 2009.

3. M. Iqbal, A. I. Bhatti, S. Iqbal, Q. Khan and I. H. Kazmi, "Parameter Estimation of Nonlinear Systems using Higher Order Sliding Modes", 7th International Conference on Control and Automation, ICCA 2009, December 9-11, Christchurch, New Zealand.

4. M. Iqbal, A. I. Bhatti, S. Iqbal, Q. Khan and I. H. Kazmi,"Fault Diagnosis of Nonlinear system using HOSM Techniques". 7th Asian Control Conference, ASCC 2009, August 27-29, 2009, Hong Kong.
5. M. Iqbal, A.I.Bhatti, S.Iqbal and Q.Khan "Parameter Estimation based Fault Diagnosis of Uncertain Nonlinear Three Tank System using HOSM Differentiator Observer" INMIC 2009.
6. M. Iqbal, A. I. Bhatti, Q. Khan, I. H. Kazmi, "Second Order Sliding Mode Observer design for Nonlinear Systems" IBCAST 2010.

1.4 Thesis Structure

This monograph is based on six chapters which are organized in the forthcoming way.

Chapter 1 gives an introduction to thesis which contains brief discussion of SMC Literature, motivation, statement of contribution and structure of the thesis.

Chapter 2 recalls the existing mathematical fundamentals which will support the research work presented in the forthcoming chapters. This chapter contains a very simple discussion about SMC, a comprehensive presentation of the theory of DSMC and a preliminary introduction of the ISMC for uncertain nonlinear systems. The subsequent work contains some comparative results which are made with the use of output derivative estimator. Therefore, it will also include the introduction of Semi High Gain Observer (SHGO) and Levant Robust Differentiator (LRD). The SHGO is used in those examples where the comparison is carried out with standard literature work.

Chapter 3 contains the first part of the contributions to this thesis. A detailed problem formulation of Single Input Single Output (SISO) uncertain nonlinear system is carried out in this chapter along with some definitions and assumptions which are either adopted from the existing literature work or established for this work. The control law is designed with strong convergence condition ([15]) which results in asymptotic sliding modes. The main claim of this chapter is that the performance is improved, chattering is attenuated and the robustness is enhanced. The proposed methodology's claim is verified by the simulation results of a couple of numerical examples.

Chapter 4 contains the extended form of the contributing work presented in chapter 3 to a case where the system is operating under a class of states matched and unmatched uncertainties. The sliding mode is enforced, in finite time, along the integral manifold using the famous reachability condition [3]. By applying the proposed controller, the system output is regulated to zero even in the presence of the uncertainties. In addition, the proposed control law is theoretically analyzed using Lyapunov Energy function and its performances is demonstrated in simulation.

Chapter 5 is based on the extension of the work, proposed in Chapter 3 and 4, to Multi Input Multi Output nonlinear systems. In first case, the problem is formulated and the control law design is developed with strong reachability condition ([16]) which, once again, results in asymptotic sliding modes and the performance is demonstrated in simulation.

In the other case, the nonlinear system is considered to be operating under matched and unmatched uncertainties. In this case the vector of outputs is regulated to zero in the uncertainty presence and the proposed control law is developed with finite time converging condition. The theoretical analysis is carried out with the introduction of a couple of theorems. A numerical example is simulated and the claim of uncertainties compensation is verified.

Chapter 6 contains the conclusion of the thesis and some suggestions are given for the future work which may be carried out with theoretical development and experimental implementations.

1.5 Summary

This chapter has provided the overview of this manuscript. The next chapter is dedicated to the demonstration of some of the fundamental mathematical essentials which will provide sufficient background for the contributing chapters.

Chapter 2

MATHEMATICAL PRELIMINARIES

Many physical systems require robustness and accurate tracking in control perspectives. Conventional nonlinear control techniques do not remain robust in the presence of internal and external disturbances. Therefore, SMC becomes a good candidate for the control of these systems. Since, SMC suffers from the worse chattering phenomena when sliding mode is established and shows low robustness to uncertainties of matched nature in reaching phase. In this thesis, a control design strategy is proposed which preserves the key features of SMC. Therefore, introduction to SMC and two of the variants of SMC the so-called Dynamic Sliding Mode Control (DSMC) and Integral Sliding Mode Control (ISMC), which are prerequisites for this thesis, are discussed in this chapter. In addition, some fundamental mathematical essentials which will provide a background for the subsequent chapters are discussed. The design methodologies of these techniques are also included briefly. Furthermore, an introduction to Semi High Gain Observer (SHGO) is recalled which is used in the simulation of a couple of examples to provide the standard results of literature. This chapter is organized as follows. An introduction to SMC is given in Section 2.1 and the design scheme of DSMC is elaborated in Section 2.2. ISMC design architecture and Output Differentiator theory is introduced in Section 2.3 and 2.4, respectively. Finally, the chapter is summarized in Section 2.5.

2.1 Sliding Mode Control

Variable Structure Control (VSC) with SMC has its own importance in the field of control engineering. VSC came into existence in the early 1950's in Soviet Union by Emelyanov and his co-researchers [23], [24]. They carried out their analysis on a second order linear system in phase variable form. Consequently, a switching control law, the so-called Variable Structure Control was proposed which showed fruitful results in comparison to simple state feedback control law. Since its introduction, it turned into a general design method for systems of different classes including uncertain nonlinear systems, large scale and infinite dimensional systems etc. The key feature of SMC is the enforcement/attraction of system's states trajectories onto a defined surface (manifold). This is called attraction/reaching phase. Traditionally, these manifolds are constructed as some hyper surface or intersection of hyper surfaces in the state space which are termed as "switching surfaces". The beauty lies in the fact that once the system states reach the switching surface, the structure of the feedback loop is adaptively altered to slide the system states along the switching manifold ([17]). This phase is called sliding phase. Having achieved the system in sliding phase, the response of the system depends, thereafter, on the gradient of the switching manifold and remains insensitive to internal parametric variations and unmodeled dynamics and external disturbances. This may be suitable to say that sliding mode control is a constrained motion. This constrained motion is termed as Sliding Mode. A system in sliding mode evolves with $n-m$ number of states with n being the dimension of the system's states and m the dimension of the control inputs. This order reduction provides invariance to the plant parameter variations and the disturbances which is one of the main benefit of SMC. In addition, it provides the decoupling of high dimensional problems into subtask of lower dimensionality. Even with provision of such interesting and key features, the imperfections in switching devices and delays results in high frequency motion the so-called chattering. In this high frequency motion the system trajectory crosses the switching manifold rather than remaining on it ([25]).

In order to have fast response, the amplitude of the switching increases across the manifold which may be dangerous to the system and the actuator health.

The reduced amplitude of the control law switching against the manifold can be achieved at the cost of the performance and robustness degradation. On the other hand, the improved performance and robustness results in the high amplitude of control law. This trade-off problem among chattering, performance and robustness attracted a number of researchers. The efforts of these researchers gave birth to different variants of the SMC technique.

Since, DSMC and ISMC are among the techniques introduced to address the chattering attenuation and robustness enhancement, respectively. DSMC is an output feedback methodology which provides control input with reduced chattering and required robustness. On the other hand, ISMC scheme is capable of providing needed robustness from the start of the process. In addition, the chattering can also be reduced with the use of a low pass filter. The subsequent sections include the design strategies of these control techniques.

2.2 Dynamic Sliding Mode Control

With the advent of differential algebraic theory, one of the most remarkable formulation of linear and nonlinear control system theory the so-called Local Generalized Controllable Canonical (LGCC) form of Fliess [26] greatly improved the applicability of SMC strategy. This has eliminated some existing disadvantages along with the enrichment of robustness properties. DSMC scheme ([14] [27]) is based on this LGCCF form which provides satisfactory results. This methodology is elaborated in the subsequent study.

Consider an n dimensional Single Input Single Output (SISO) nonlinear system with the following state space representation:

$$\dot{x} = f(x,t) + g(x,t)u \qquad \textit{Eq (2.1)}$$

$$y = h(x, t) \qquad \textit{Eq (2.2)}$$

where $x \in R^n$ is the states vector and $u \in R$ is the scaler control input. f: $R^n \times R^+ \rightarrow R^n$ and g: $R^n \times R^+ \rightarrow R^n$ and h:$R^n \times R^+ \rightarrow R$ are smooth vector fields. To have a proper statement of the problem, it is assumed that the output is continuously differentiable with respect to some defined relative degree.

The relative degree is the order of differentiation of the output with respect to the nonlinear dynamics of the system in which the control input appears explicitly. In other words, one can say that some type of nonlinear transformation exists which are input and state dependent. The input output representation is obtained by taking successive time derivatives of the output with respect to the given system. The d^{th} time derivative of the output function $h(x, t)$ with respect to Eq (2.1) is defined as

$$y^{(d)} = \phi(y, \dot{y}, \ddot{y}, \cdots, y^{(d-1)}, u, \dot{u}, \ddot{u}, \cdots, u^{(\beta)}) \qquad \textit{Eq (2.3)}$$

Now, by defining $y^{(i-1)} = \xi_i$ for $i = 1, 2, \cdots, d$, the system in Eq (2.3), can be written as

$$\dot{\xi}_1 = \xi_2$$

$$\dot{\xi}_2 = \xi_3$$

$$\vdots$$

$$\dot{\xi}_d = \phi(\hat{\xi}, \hat{u}) \qquad \textit{Eq (2.4)}$$

where $\hat{y} = [y, \dot{y}, \cdots, y^{(d-1)}]^T = \hat{\xi} = [\xi_1, \xi_2, \cdots, \xi_d]^T$ and $\hat{u} = [u, \dot{u}, \cdots, u^{(\beta)}]^T$. The variables $\xi_i, i = 1, 2, \cdots, d$ are referred to as generalized phase variables. If $d < n$, then the state realization in Eq (2.1) is termed as minimal realization [14]. To deal with a very simple structure, it is further assumed that the nonlinear system is called square system if the number of inputs to the system is equal to the number of outputs.

The coordinate transformation from x-variables to generalized phase variables is a full rank map which is defined by [14]

$$\xi = T(x, u, \dot{u}, \ddot{u}, \cdots, u^{(\beta)}) = \begin{bmatrix} h(x) \\ \dot{h(x)} \\ \vdots \\ h^{(r)}(x, u) \\ \vdots \\ h^{(n-1)}(x, u, \dot{u}, \cdots, u^{(\beta-1)}) \end{bmatrix} \qquad \textit{Eq (2.5)}$$

This nonlinear transformation defined in Eq (2.5), transforms Eq (2.1) into Eq (2.4). The design of the control algorithm can be worked out after the satisfaction of the following definition.

Definition 2.1. The system in Eq (2.4) is named as proper [15], if

1. $\phi(\hat{\xi}, \hat{u}) \in C^1$
2. the regularity condition $\frac{\partial \phi(\hat{\xi}, \hat{u})}{\partial u^{(\beta)}} \neq 0$ holds for some defined $\hat{\xi}$ in the neighborhood of some operating point.

Definition 2.2. The system in Eq (2.4) is called minimum phase if the following defined dynamics are uniformly asymptotically stable [15].

$$\phi(0, \hat{u}, t) = 0 \qquad \textit{Eq (2.6)}$$

If the Definition 2.2 is not satisfied then the system may be categorized either as weakly minimum phase system or non-minimum phase system. The dynamics presented in *Eq (2.6)* are the zero dynamics of the control inputs. It is different from the zero dynamics mentioned in [28] which are the dynamics of uncontrollable states.

2.2.1 Control Design Methodology

The traditional methodologies of SMC comprises of two steps which may be categorized as the selection/definition of sliding surface and the controller design. DSMC strategy is also being supported by the above two phase.

2.2.1.1 Sliding Surface Design

The sliding surface being used in DSMC is similar to the conventional sliding surface with only one noticeable difference. The states in this sliding surface are the output and its derivatives or in other words, one may say the variable appearing in the LGCCF form are linearly combined to design the sliding surface. These sliding surface must be selected according to the response of the system. In literature of DSMC, two types of sliding surfaces are used for design.

- Direct Sliding Surface

The direct sliding surface is based on only the variables of I-O forms (or the Variable appearing in the LGCC forms). Mathematically, it can be defined as [15]

$$s(\hat{\xi}) = \sum_{i=1}^{n} c_i \xi_i \qquad \textit{Eq (2.7)}$$

where with $c_n = 1$. This is called Hurwitz polynomial which is furnished in [14] and [27].

- Indirect Sliding Surface

The indirect sliding surface includes the generalized variables and the nonlinear function which is being defined as a function of the phase variables and the control inputs. This may be expressed as

$$s(\hat{\xi}) = \sum_{i=1}^{n} c_i \xi_i + \phi(\hat{\xi}, \hat{u}, t) \qquad \textit{Eq (2.8)}$$

The focus of a control engineer is to have a control law which meets all the requirements as needed. One of these specification is robustness. In order to have robustness, a strong reachability condition is defined as follows (see, [14], [15]).

Definition 2.3. A sliding reachability condition

$$\dot{s}(\hat{\xi}) = -\gamma(s) \qquad \textit{Eq (2.9)}$$

is called strong if

1. $\gamma(0) = 0$
2. $\gamma(s) \in C^{(0)}$ function of s if $s \neq 0$.
3. $\gamma(s)$ is bounded for all points in the neighborhood of the origin.
4. $s\gamma(s) > ks^2$ if $s \neq 0$ for some $k > 0$.

This reachability condition will ensure the asymptotic sliding modes enforcement and the states will also reach the origin asymptotically.

2.2.1.2 Control Design

The time derivative of *Eq (2.7)* along the trajectories of Eq (2.4), yields

$$\dot{s}(\hat{\xi}) = \sum_{i=1}^{n-1} c_i \xi_{i+1} + \phi(\hat{\xi}, \hat{u}, t) \qquad \textit{Eq (2.10)}$$

Now, by comparing *Eq (2.9)* and *Eq (2.10)*, one obtains

$$u^{(\beta)} = \alpha(\hat{\xi}, u, \dot{u}, \cdots, u^{(\beta-1)}) \qquad \textit{Eq (2.11)}$$

The right hand side of *Eq (2.11)* is a discontinuous function of output and their derivatives and possibly the control input and their derivatives. The discontinuity of *Eq (2.11)* is caused by the sign function. This control law will provide asymptotically stable motion to the origin of the generalized phase coordinate system. In addition, the control law provides a system which is controllable canonical form in phase variable. The response of the system's states convergence, when sliding mode is established, depends on gradient of the sliding variable defined in Eq (2.7). To clarify the above techniques, the following example is considered.

2.2.2 Example

Consider the following nonlinear system adopted from [29]

$$\dot{x_1} = -x_1 + \exp(2x_2)$$

$$\dot{x_2} = 2x_1x_2 + sin(x_2) + \frac{1}{2}u \qquad \textit{Eq (2.12)}$$

$$\dot{x_3} = 3x_2$$

The output of this system is $y = h(x) = x_3$. The relative degree of this system with respect to the output function is 2. Consequently, the controller will bear one time derivative. The system in LGCCF can be illustrated as follows

$$\dot{\xi_1} = \xi_2$$

$$\dot{\xi_2} = \xi_3$$

$$\dot{\xi}_3 = \phi(\hat{y}, \hat{u}) \qquad Eq\ (2.13)$$

where $\phi(\hat{y}, \hat{u}) = -4x_1x_2 + (4x_1 + 2cos(x_2))(2x_1x_2 + sin(x_2) + \frac{1}{2}u) + \dot{u}$. The transformations used here are $\hat{y} = [y, \dot{y}, \ddot{y}] = \hat{\xi} = [\xi_1, \xi_2, \xi_3]^T = [x_3, 2x_2, 2(2x_1x_2 + sin(x_2) + \frac{1}{2}u)]^T$. It is obvious that the definition 2.1 and 2.2 holds. Now, the sliding surface is defined by

$$s(\hat{\xi}) = c_1\xi_1 + c_2\xi_2 + \xi_3 \qquad Eq\ (2.14)$$

The time derivative of *Eq (2.14)* along the dynamics of *Eq (2.13)*, yields

$$\dot{s}(\hat{\xi}) = c_1\xi_2 + c_2\xi_2 + \phi(\hat{y}, \hat{u}) \qquad Eq\ (2.15)$$

Keeping in view the definition 2.3, a strong decoupled reachability condition can be defined by

$$\dot{s} = -K_1(s + W signs) \qquad Eq\ (2.16)$$

Comparing *Eq (2.15)* and *Eq (2.16)* and simplifying, one has

$$\dot{u} = -[(c_1(2x_2) + c_2(2x_1x_2 + sin(x_2) + \frac{1}{2}u) - 4x_1x_2 + (4x_1 + 2cos(x_2)) \qquad Eq\ (2.17)$$

$$\times(2x_1x_2 + sin(x_2) + \frac{1}{2}u) + K_1(s + W signs)]$$

This controller can be used to control the plant output by filtering the control input once. This filtration results in a reduced chattering of the control input. The robustness of this methodology also depends on the definition of the reachability condition (see for detail, [14], [27], [15] & [16]).

2.3 Integral Sliding Mode Control

Integral sliding mode attempts to reject uncertainties and may also be used to avoid chattering [17]. It is the traditional sliding mode with a difference of sliding manifold and independence of reaching phase (a necessary part of the conventional sliding modes). It means that sliding occurs from the initial time instant. The system operates with full states in integral sliding mode while in simple sliding mode the system has reduced order dynamics [17]. The simple introduction of integral sliding mode is discussed below. Consider the following nonlinear system with state space description

$$\dot{x} = f(x,t) + B(x,t)u \qquad \text{Eq (2.18)}$$

where $x \in R^n$ is the state vector and $u \in R$ is the control which appears linearly in the systems representation. To proceed to the control design, it is assumed that

1. B(x,t) is full rank or the system is controllable.
2. The system in ideal case operates under a feedback control law $u = u_0(x)$

Thus the system operating under u_0 may have the following form

$$\dot{x}_0 = f(x_0,t) + B(x_0,t)u_0 \qquad \text{Eq (2.19)}$$

with x_0 representing the state trajectory under u_0. However, in practical applications, the system operates under uncertainties caused by unmodeled dynamics, parametric variations and external disturbances. Thus the system *Eq (2.18)* takes the form

$$\dot{x} = f(x,t) + B(x,t)u + \zeta(x,t) \qquad \text{Eq (2.20)}$$

where $\zeta(x,t)$ represent the perturbations due to uncertainty in dynamics which may be caused by parameter variations and external disturbances. Furthermore, the uncertainties are termed as matched uncertainties if they affect the system exactly at that point where the control input is applied to the system. Mathematically, it can be expressed by the following equivalent forms

$$\zeta(x,t) = B(x,t)\delta \qquad \text{Eq (2.21)}$$

Therefore, it is also assumed that the uncertainties appearing in the system are norm bounded. i.e., $||\zeta(x,t)|| \leq \zeta^{+}(x,t)$, with $\zeta^{+}(x,t)$ being some known positive scalar function.

The objective is to design a control law which meets $x(t) \equiv x_0(t)$ from the initial time instant $x(0) = x_0(0)$. The required control law is of the nature

$$u = u_0 + u_1 \qquad \textit{Eq (2.22)}$$

where u_0 being the ideal control and u_1 is designed to reject perturbation term $\zeta(x,t)$. The use of *Eq (2.22)* in *Eq (2.20)*, yields

$$\dot{x} = f(x,t) + B(x,t)u_0 + B(x,t)u_1 + \zeta(x,t) \qquad \textit{Eq (2.23)}$$

Now, the sliding manifold is defined as [4]

$$\sigma(x) = \sigma_0(x) + z \qquad \textit{Eq (2.24)}$$

The first term in the right hand side of *Eq (2.24)* indicates the contribution of conventional sliding surface and the second term is the integral term which is to be determined in the subsequent analysis. The time derivative of *Eq (2.24)* along the dynamics of *Eq (2.23)*, takes the form

$$\dot{\sigma} = \nabla\sigma_0 \left[f(x,t) + B(x,t)u_0 + B(x,t)u_1 + \zeta(x,t)\right] + \dot{z} \qquad \textit{Eq (2.25)}$$

Now, selecting the integral term dynamics of the form

$$\dot{z} = -\frac{\partial\sigma_0(x,t)}{\partial x}\left(f(x,t) + B(x,t)u_0\right) \qquad \textit{Eq (2.26)}$$

$$z(0) = -\sigma_0(x(0))$$

where the initial condition $z(0)$ is chosen to satisfy the requirement $\sigma(0) = 0$. The satisfaction of this condition certify the occurrence of sliding mode at the starting of the process. The above choice of the integral term dynamics reduces the *Eq (2.25)* to the following form

$$\dot{\sigma} = \nabla\sigma_0 \left[B(x,t)u_1 + \zeta(x,t)\right] \qquad \textit{Eq (2.27)}$$

In order to achieve the congruence condition, $x(t) \equiv x_0(t)$, adapting the procedure of the equivalent control method [3]. The expression of u_{1eq} becomes

$$u_{1eq} = -\delta \qquad Eq\ (2.28)$$

The justification of the smart condition $u_{1eq} = -\delta$, leads to the forthcoming state equations which governs the motion of the system in sliding mode.

$$\dot{x} = f(x,t) + B(x,t)u_0 \qquad Eq\ (2.29)$$

To enforce the sliding mode along the integral sliding manifold *Eq (2.24)*, the discontinuous control function u_1 in *Eq (2.22)* may be selected with the following expression

$$u_1 = -M(x)sign(\sigma) \qquad Eq\ (2.30)$$

The above analysis can be carried only when $det[\nabla\sigma_0 B(x,t)] \neq 0$ and the positive scalar function $M(x)$ can be designed such that the norm of $M(x)$ must be greater than or equal to the norm of the uncertain terms. This methodology provides robustness from the very beginning of the process due to the elimination of the reaching phase with acceptable performance. However, chattering can be reduced with the use of some low pass filter. The design methodology is elaborated with the forthcoming example.

2.3.1 Example

Consider a constant length pendulum problem with the following dynamical equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}sin(x_1) + \frac{1}{ml^2}u \end{aligned} \qquad Eq\ (2.31)$$

where $m = 1$ is the mass, $l = 1$ is the length of the pendulum and $g = 9.81$ is the gravitational acceleration. The initial conditions are set to $x_1(0) = 0$ and $x_2(0) = 0$. The control objective is to design a integral sliding control law which steer the states of this system to origin. The control law can be designed according to aforementioned procedure. The control law is composed of two components which appears as follows

$$u = u_0 + u_1 \qquad \textit{Eq (2.32)}$$

where u_0 is the continuous component and u_1 is the discontinuous component. The continuous component can be designed by pole placement while assuming that $sin(x_1) = 0$ i.e., $u_0 = -k_1x_1 - k_2x_2$, where k_1, k_2 are the gains of the controller. The design of the discontinuous term can be carried out by defining the following integral manifold

$$\sigma(x_1, x_2) = c_1x_1 + x_2 + z \qquad \textit{Eq (2.33)}$$

Now, taking the derivative of *Eq (2.33)* along *Eq (2.31)*, one has

$$\dot{\sigma} = c_1x_2 + (\frac{-g}{l}sin(x_1) + \frac{1}{ml^2}(u_0 + u_1)) + \dot{z} \qquad \textit{Eq (2.34)}$$

Choosing $\dot{z} = -(c_1x_2 + (\frac{-g}{l}sin(x_1) + \frac{1}{ml^2}u_0))$ with $z(0) = 0$, the above equation *Eq (2.34)* becomes

$$\dot{\sigma} = \frac{1}{ml^2}u_1 \qquad \textit{Eq (2.35)}$$

Comparing *Eq (2.35)* with $\dot{\sigma} = -K_1sign(\sigma)$, one has

$$u_1 = -Ksign\sigma \qquad \textit{Eq (2.36)}$$

where $K = (ml^2)K_1$ is the gain of the discontinuous component. The final control law can be obtained by substituting the designed expression of continuous and discontinuous components in *Eq (2.32)*. This control law eliminate the reaching phase and results in the robust regulation of the states to the origin. In the coming section, an observer designed is outlined which provides the estimates of the states of the systems.

2.4 Output Differentiator or Observer

In conventional SMC, it is assumed that either all states are available or some part of the states is available. If all the states are defined to be available then a control algorithm is designed for the stabilization of the plant based on these states. If some part of these states is not available directly in application, then a state observer based control is designed for the control of the plant. In output feedback control methodologies, the output and its derivatives are needed to design a stabilizing controller. The output's derivatives estimation is studied by [30], [31] but the semi high gain observer used in [15], [16] and Levant Robust differentiator (LRD) [13] are often used. In the forth coming chapters, the semi high gain observer is used to provide the derivatives. In the subsequent subsections both the SHGO and LRD methodologies are briefly discussed.

2.4.1 Semi High Gain Observer

The research work presented in the forthcoming chapters needs the accurate estimate of the outputs and its derivatives in real application. Note that, the motivation of the use of semi high gain observer is based on two facts. This provides fast convergence to the actual values but the problem with SHGO is the peaking phenomenon associated to high gains. The other fact is that the existing work of the literature ([15], [16]) is carried out with the use of this semi high gain observer. Therefore, its presentation is suitable in this monograph.

Consider a nonlinear system of the form

$$\dot{\xi}_i = \xi_{i+1}, i = 1, 2, ..., n-1$$

$$\dot{\xi}_n = \varphi(\xi, z, t) \qquad \textit{Eq (2.37)}$$

where ξ is the state vector, z is control input and $\varphi(\xi, z, t)$ represents some function of these variables. The function $\varphi(\xi, z, t)$ is called locally Lipschitz if there exists $L \geq 0$ such that

$$|\varphi(\xi, z, t) - \varphi(x, z, t)| \leq L|\xi - x| \qquad \textit{Eq (2.38)}$$

uniformly with respect to ξ and x, where x is state of the observer, in their respective neighborhoods for all times. The parameter L will depend on the radii of these neighborhoods [15]. Assume that only ξ_1 is available. Now, consider the observer of the form

$$\dot{x}_1 = x_2 + \frac{\alpha_1}{\epsilon}(\xi_1 - x_1)$$

$$\dot{x}_2 = x_3 + \frac{\alpha_2}{\epsilon^2}(\xi_1 - x_1)$$

$$\vdots \qquad \textit{Eq (2.39)}$$

$$\dot{x}_{n-1} = x_n + \frac{\alpha_{n-1}}{\epsilon^{n-1}}(\xi_1 - x_1)$$

$$\dot{x}_n = \varphi(x, v, t) + \frac{\alpha_n}{\epsilon^n}(\xi_1 - x_1)$$

This can also be expressed as [15]:

$$\dot{x} = E_n x + \alpha(\epsilon)(\xi_1 - x_1) + D_n \varphi(x, v, t)$$

where $\alpha(\epsilon) = \left[\frac{\alpha_1}{\epsilon}, \frac{\alpha_2}{\epsilon^2}, \cdots, \frac{\alpha_n}{\epsilon^n}\right]$ and ϵ is bounded semi-high gain i,e., $0 < \epsilon < \bar{\epsilon}$. Furtheremore, $E_n = \begin{bmatrix} O_{(n-1)\times 1} & I_{(n-1)\times(n-1)} \\ O_{1\times 1} & O_{1\times(n-1)} \end{bmatrix}$ and $D_n = \begin{bmatrix} O_{(n-1)\times 1} \\ I_{1\times 1} \end{bmatrix}$. This boundedness reflects the fact that the fast convergence of the observer to zero confirms the stability of the closed loop control system (see for detail, [15] and the reference related to SHGO). The difference between this semi high gain observer and the traditional high gain observer is that gain parameter ϵ does not necessarily approach to zero to ensure the asymptotic stability of the whole closed loop system.

2.4.2 Levant Robust Differentiator

Another approach is the robust exact differentiator used to estimate the output derivatives in the presence of infinitesimal Lebesgue-measurable measurement noises provided the second derivative of the output is bounded. The motivation behind this presentation is that the subsequent differentiator works very well in the presence of measurable noises. Its use is very suitable in HOSM controller where robustness is needed. An n times successively implemented differentiator will provide an accuracy of the order of $\epsilon^{2^{-n}}$ [13]. The accuracy of the differentiator degrades with the increase of the order of differentiation. Let the Lipschitz constant L be the bound of the derivative of the output. The best possible accuracy of the differentiator is proportional to $L^{i/(n+l)}\epsilon^{(n+l-i)/(n+l)}, i = 0, 1, \cdots, n.$. The mechanism of the differentiator is outlined in the forthcoming discussions.
Consider a function of unknown features defined on a semi infinite domain with bounded Lebesgue measurable noise. Let $f_0(t)$ be the base signal with bounded

derivatives being bounded by the Lipschitz constant $L > 0$. The objective is to find out the real estimates $\dot{f}_0(t), \ddot{f}_0(t), \cdots, f_0^n(t)$ in the presence of uncertainties. A simple scheme is presented here

$$\dot{z}_0 = v_0, v_0 = -\lambda_0|z_0 - f(t)|^{n/(n+l)} sign(z_0 - f(t)) + z_1$$

$$\dot{z}_1 = v_1, v_1 = -\lambda_1|z_1 - v_0|^{(n-1)/n} sign(z_0 - v_0) + z_2$$

$$\vdots \qquad \textit{Eq (2.40)}$$

$$\dot{z}_{n-1} = v_{n-1}, v_{n-1} = -\lambda_{n-1}|z_{n-1} - v_{n-2}|^{1/2} sign(z_{n-1} - v_{n-2}) + z_n$$

$$\dot{z}_n = -\lambda_n sign(z_n - v_{n-1})$$

This differentiator exactly estimates the derivatives which is widely used in the implementation of HOSMC scheme. There were a number of shortcomings in this differentiator which were removed in the extended work of Levant [32]. This algorithm can be explained via the following example.

2.4.2.1 Example

Consider the example referred in *Eq (2.31)*. Assume that only the position x_1 is available. The velocity can be estimated by the above Levant differentiator as follows

$$\dot{Z}_0 = v_0, v_0 = -\lambda_0|Z_0 - x_1|^{1/2} sign(Z_0 - x_1) + Z_1$$

$$\dot{Z}_1 = v_1, v_1 = -\lambda_1| sign(Z_0 - v_0)$$

where λ_0 and λ_1 are the gains of the differentiator. The terms Z_0, Z_1 are the exact estimates of x_1 x_2, respectively. The initial conditions of the estimator(differentiator) were set to $Z_0 = x_1(0) - 0 = 0$ and $Z_1(0) = 0$ and the control law based on these estimates becomes

$$u = -k_1 Z_0 - k_2 Z_1 - K sign(c_1 Z_0 + Z_1 + \hat{z})$$

where $\hat{z}$ is the integral term of the integral manifold which depends on the estimated position and velocity of the system. The design of the control law and the differentiator can be carried out separately and then they can be implemented on the actual system.

2.5 Summary

This chapter introduced the fundamentals of the theory of DSMC, ISMC and Output Derivative Estimator (differentiator). The DSMC design frame work is introduced and a very short understandable design method of the existing ISMC is included in this chapter. The main purpose of these two techniques is to provide the reader a background which will be helpful in the next chapters. The output feedback methodologies often needs derivative estimator in application, therefore, a SHGO is also presented in this chapter.

In the next chapter, an output feedback control methodology is proposed for non-linear systems. The problem is suitably transformed and a dynamic control law composed of a continuous and discontinuous terms is designed and the simulation results for a couple of case studies are given.

Chapter 3

DYNAMIC INTEGRAL SLIDING MODE CONTROL METHODOLOGY FOR NONLINEAR SISO SYSTEMS

In real applications, as already discussed in chapter 2, a wide class of nonlinear systems often need robustness against internal and external disturbances. The conventional SMC, which ensures good robustness when sliding mode is established, suffers from the chattering phenomenon against the sliding set. This chattering phenomenon is often needed to be reduced or eliminated using different approaches. These include the most famous boundary layer approach (see for instance, [4], [29]). These approaches resulted in chattering alleviation. However, the robust performance was certainly weakened with this boundary layer because the robust performance and reduced chattering are inversely related with maximum dependence on the choice of the boundary layer thickness. As it is well known that, in most of the dynamic systems, the undesirable threatening phenomena (chattering) is caused by switching action with the parasitic dynamics. An asymptotic estimator based strategy is used to build a high frequency by pass loop ([33] and [34]) which localizes the high frequency phenomenon in feedback loop. This feedback loop is accomplished with a discontinuous control loop which is closed through the observer of the plant (see for detail, [4], [35]). The asymptotic observer's results in better estimates of the system's behavior than the plant. Consequently, the discontinuous controller switching against the observer's states (with smaller imperfections) results in chattering inside a high frequency loop which by-passes the plant. This approach needs that the observer must converge to zero state error asymptotically. The HOSM technique (see for instance, [7], [9], [8], [36], [37], [38], [39], [32],, [40], [41], [42]) are advanced SMC techniques. The main objective of this control approach was to provide a control law, while preserving the important properties, with either chattering free or with reduced chattering along with the relaxation in the relative degree one requirement. DSMC (see for instance, [15], [16], [27], [43], [44], [45]) was also established to eliminate chattering with the enforcement of asymptotic sliding modes along the sliding surface.

Note that, all of the above discussed approaches excellently removed chattering phenomena in sliding mode. In addition, when sliding mode is established, the nonlinear system becomes robust against the uncertainties and disturbance of the so-called matched nature (see for definition [46]). However, in many applications, the systems don't remain robust against the matched uncertainties in the reaching phase. Therefore, the ISMC approached was developed (see for instance, [4], [47], [48]) to eliminate the reaching phase and to enhance robustness against the matched uncertainties from the very beginning. This technique brought revolution in the theory of SMC and it showed good results in different applications (see for instance, [4], [49], [50]). So in this context, in this chapter, DSM control approach is synthesized with the ISM approach for SISO nonlinear systems. The namely proposed technique DISMC inherits the good features of both the established techniques in the form of reaching phase elimination and chattering alleviation, robustness enhancement and performance improvement. This chapter is organized as follows. In the Section 3.1, the problem formulation is presented and in Section 3.2 the design of the control law is presented. A couple of numerical examples are presented in Section 3.3, one relevant to relative degree two, the other relevant to relative degree three. A comprehensive comparative analysis is carried out with some standard results in the example 1, while in the example 2 a comparative study of DSMC and DISMC is carried out and the control law robustness is checked in the presence of some uncertainties. In the last section the chapter is summarized.

3.1 Problem Formulation

Consider a SISO nonlinear system described by the state equation

$$\dot{x} = f(x) + g(x)u + \zeta(x,t) \qquad \textit{Eq (3.1)}$$

$$y = h(x) \qquad \textit{Eq (3.2)}$$

where $x \in R^n$ is the measurable state vector, $u \in R$ is scalar control input, $f(x)$ and $g(x)$ are sufficiently smooth vector fields, $\zeta(x,t)$ represents the uncertainties. These uncertainties occur due to unmodeled dynamics and parametric variations and $h(x)$ is a measurable scalar output function. The uncertain function $\zeta(x,t)$

has the characteristics of norm boundedness by a scalar function i.e., $|\zeta(x,t)| \leq \zeta_0$, where ζ_0 is some positive constant or scalar known function.

The problem we want to solve (Problem 1) is that of steering the output y to zero asymptotically, i.e., an output regulation problem is considered here in the presence of some uncertainties of matched nature. In order to design the proposed control law, a very simple and understandable discussion on the output feedback technique is presented in the forthcoming study. To this end, we define the derivative of the output function $h(x)$ in the direction of the vector field $f(x)$ as follows [28].

$$Ł_f h(x) = \partial h(x)/\partial x f(x) = \nabla h(x) f(x)$$

Recursively it can be defined as

$$Ł_f^0 h(x) = h(x)$$

$$L_f^j h(x) = L_f(L_f^{(j-1)} h(x)) = \nabla(L_f^{(j-1)} h(x)) f(x), j = 1, 2, ...$$

The relative degree r of the system with respect to the output is the r^{th} derivative of the output function in which the input appears explicitly (see, [28] and [29]), one has

$$y^{(r)} = L_f^r h(x) + L_g(L_f^{(r-1)} h(x))u + \zeta(x,t) \qquad \textit{Eq (3.3)}$$

Subject to the following conditions

1. $L_g(L_f^{(i)} h(x)) = 0$ for all x in the neighborhood of x_0 for $i < r-1$.
2. $L_g(L_f^{(r-1)} h(x)) \neq 0$

Two possible cases arise for *Eq (3.3)*:
If $r = n$, then this becomes a trivial case which shows that the system is already in canonical form. This case is neglected in this work.
If $r < n$, then, there exist some positive integer k which satisfy $r + k = n$. Then taking the successive k derivatives of *Eq (3.3)*, one obtains the n^{th} derivative of output function as follows

$$\begin{aligned} y^{(n)} = {} & L_f^n h(x) + L_g(L_f^{(n-1)} h(x))u + \cdots + L_f L_g L_f^{(r-1)} h(x) u^{(k-1)} \\ & + L_g^2 L_f^{(r-1)} h(x) u u^{(k-1)} + L_g L_f^{(r-1)} h(x) u^{(k)} + \zeta^*(x, u, \dot{u}, \ldots, u^{(n-1)}, t) \end{aligned} \qquad (3.4)$$

where $\zeta^*(x, u, \dot{u}, \ldots, u^{(n-1)}, t)$ is the norm bounded function which represents the uncertainties and their time derivatives. The constant k shows the number of differentiation of *Eq (3.3)*. Now, suppose that

$$\varphi(\hat{y}, \hat{u}) = L_f^n h(x) + L_g(L_f^{(n-1)} h(x))u + \cdots + L_f L_g L_f^{(r-1)} h(x) u^{(k-1)} + L_g^2 L_f^{(r-1)} h(x) u u^{(k-1)})$$

and

$$\gamma(\hat{y}) = L_g L_f^{(r-1)} h(x)$$

where $\hat{y} = [y, \dot{y}, \cdots, y^{(n-1)}]^T$ and $\hat{u} = [u, \dot{u}, \cdots, u^{(n-1)}]^T$. Now by defining the transformation $y^{(i-1)} = \xi_i$ for $i = 1, 2, \cdots, n$ and $\hat{y} = \hat{\zeta}$, the system *Eq (3.4)* can be written as follows [14]

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_n &= \varphi(\hat{\xi}, \hat{u}) + \gamma(\hat{\xi}) u^{(k)} + \zeta^*(\hat{\xi}, t) \\ &= \phi(\hat{\xi}, \hat{u}, u^{(k)}) + \zeta^*(\hat{\xi}, t) \end{aligned} \qquad Eq\ (3.5)$$

This is exactly the LGCC form and $\zeta^*(\hat{\xi}, t)$ is the uncertainty.

Assumption 1. *let $\zeta^*(\hat{y}, t)$ and its derivatives be bounded and must satisfy:*

$$|\zeta^*(\hat{\xi}, t)| \leq K_1$$

where K_1 is the uncertainty bound in [5].

A wide class of nonlinear systems can be put into input output (I-O) form with the addition of compensator term which appears as a chain of integrators [26]. Now, the nominal system corresponding to system *Eq (3.5)* can be obtained by replacing $\zeta^*(\hat{\xi}, t) = 0$ which is given by .

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \end{aligned}$$

$$\dot{\xi}_n = \varphi(\hat{\xi}, \hat{u}) + \gamma(\hat{\xi})u^{(k)} \qquad \textit{Eq (3.6)}$$
$$= \phi(\hat{\xi}, \hat{u}, u^{(k)})$$

Assumption 2. *The system Eq (3.6) is proper and minimum phase according to Definitions 2.1 and 2.2.*

Now, the original problem (Problem 1) can be reformulated with reference to *Eq (3.5)* under Assumptions 1 and 2 and to the nominal system in *Eq (3.6)*. The new problem (Problem 2) is that of steering the state vector $\hat{\xi} = [\xi_1, \xi_2, ..., \xi_n]^T$ of system *Eq (3.5)* to zero asymptotically. Now, the solution to Problem 2 is a clear solution to Problem 1, since $y = \xi_1$. This completes the problem formulation. In the next section, the control law design is presented.

3.2 Control Law Design

In traditional SMC and DSMC methodologies the control law has only one discontinuous term. However, in the proposed control technique the control law is dynamic and it contains two dynamic terms which appear with the following mathematical expression:

$$u^{(k)} = u_0^{(k)} + u_1^{(k)} \qquad \textit{Eq (3.7)}$$

The first part $u_0^{(k)} \in R$, is continuous which is used to stabilize the nominal system asymptotically when sliding mode is established. The second part $u_1^{(k)} \in R$ is discontinuous in nature called the dynamic integral control which efficiently rejects the uncertainties. These uncertainties may be due to external disturbances or internal parametric uncertainties etc. Note that, the proposed control law design is carried out while ignoring the uncertainties. In the subsequent subsections, the design of these components is demonstrated.

3.2.1 Design for Continuous Component

To facilitate the design of the continuous control component, the nominal system in *Eq (3.6)* can be written in alternate form as follows:

$$\dot{\xi}_1 = \xi_2$$

$$\dot{\xi}_2 = \xi_3$$

$$\vdots$$

$$\dot{\xi}_n = \chi(\hat{\xi}, \hat{u}, u^{(k)}) + u^{(k)} \qquad \textit{Eq (3.8)}$$

where $\chi(\hat{\xi}, \hat{u}, u^{(k)}) = \varphi(\hat{\xi}, \hat{u}) + (\gamma(\hat{\xi}) - 1)u^{(k)}$.

In the design of $u_0^{(k)}$, the system in *Eq (3.8)* is considered to be independent of nonlinearities i.e.,$\chi(\hat{\xi}, \hat{u}, u^{(k)}) = 0$ in the very beginning and it is also supposed that the system operates under $u_0^{(k)}$ only from the beginning of the process. Consequently, the system in *Eq (3.8)* takes the following form:

$$\dot{\hat{\xi}} = A\hat{\xi} + Bu_0^{(k)} \qquad \textit{Eq (3.9)}$$

where $A = \begin{bmatrix} O_{(n-1)\times 1} & I_{(n-1)\times(n-1)} \\ O_{1\times 1} & O_{1\times(n-1)} \end{bmatrix}$ and $B = \begin{bmatrix} O_{(n-1)\times 1} \\ I_{1\times 1} \end{bmatrix}$. The control component, $u_0^{(k)}$, is designed by the simple state feedback control design procedure of LQR. Thus, the expression of this control component becomes

$$u_0^{(k)} = -K_0^T\hat{\xi} \qquad \textit{Eq (3.10)}$$

which minimizes the quadratic cost function

$$J = \int_0^\infty \left[\hat{\xi}^T Q\hat{\xi} + R(u_0^{(k)})^2\right] dt$$

subject to the system dynamics

$$\dot{\hat{\xi}} = A\hat{\xi} + Bu_0^{(k)}$$

The control law gains vector K_0^T can be determined by the solution of the Riccati's equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

and

$$K_0 = R^{-1}B^T P$$

In the above expressions, P and Q are symmetric positive definite matrices. This completes the design of $u_0^{(k)}$.

3.2.2 Design for Discontinuous Component

The existing DSMC scheme is designed with the choice of either direct sliding surfaces or indirect sliding surface being defined in section 2.2.2. In the proposed design technique, the dynamic controller design uses an integrals manifold instead of conventional sliding surface. In order to attain the desired performance and to robustly compensate the uncertainties with reduced chattering, the dynamic controllers $u_1^{(k)}$ is formulated by first defining the integral sliding surface. The integral sliding surface is designed in such a way that the reaching phase is eliminated. The uncertainties which may result in the instability of the system in the reaching phase are handled via this elimination. The integral manifold, defined here, is analogous to that reported [17]

$$\sigma(\hat{\xi}) = \sigma_0(\hat{\xi}) + z \qquad \textit{Eq (3.11)}$$

where $\sigma_0(\hat{\xi})$ is the Hurwitz polynomial which is mathematically defined in *Eq (2.7)* by $\sigma_0(\hat{\xi}) = \sum_{i=1}^{n} c_i \xi_i$ with $c_n = 1$ and z is the integral term. The time derivative of *Eq (3.11)* along *Eq (3.8)*, yields

$$\dot{\sigma}(\hat{\xi}) = \sum_{i=1}^{n-1} c_i \xi_{i+1} + \chi(\hat{\xi}, \hat{u}, u^{(k)}) + u_0^{(k)} + u_1^{(k)} + \dot{z} \qquad \textit{Eq (3.12)}$$

Now, by choosing the dynamics of the integral term according to the forthcoming expression

$$\dot{z} = -\left(\sum_{i=1}^{n-1} c_i \xi_{i+1} + u_0^{(k)} \right) \qquad \textit{Eq (3.13)}$$

with initial conditions $z(0) = -\sigma_0(\xi(\hat{0}))$, the expression in *Eq (3.12)* reduces to

$$\dot{\sigma}(\hat{\xi}) = \varphi(\hat{\xi}, \hat{u}) + (\gamma(\hat{\xi}) - 1)u_0^{(k)} + \gamma(\hat{\xi})u_1^{(k)} \qquad Eq\ (3.14)$$

This initial condition, $z(0)$, of the integral terms is adjusted in such a way that the sliding surface start at 0 at time $t = 0$. The design of the discontinuous control law can be facilitated with the choice of some suitable sliding reachability condition. The reachabilitiy condition, for the proposed discontinuous control component, is selected according to Definition 2.3 which is given as follows

$$\dot{\sigma}(\hat{\xi}) = -K_1(\sigma + W sign(\sigma)) \qquad Eq\ (3.15)$$

By comparing *Eq (3.14)* and *Eq (3.15)*, the expression of dynamic controller $u_1^{(k)}$ becomes

$$u_1^{(k)} = -\frac{1}{\gamma(\hat{\xi})}\Big(\varphi(\hat{\xi}, \hat{u}) + (\gamma(\hat{\xi}) - 1)u_0^{(k)} + K_1(\sigma + W sign(\sigma)\Big) \qquad Eq\ (3.16)$$

This control law enforces sliding mode along the sliding manifold defined in *Eq (3.11)*. The constant K_1 is the control gain and can be selected according to uncertainty bounds [5](chapter 5), and the constant W can be defined according to application with value between $0 < W < 1$. Thus, the final control law, being obtained by substituting *Eq (3.10)* and *Eq (3.16)* in *Eq (3.7)*, becomes

$$u^{(k)} = u_0^{(k)} - \frac{1}{\gamma(\hat{\xi})}\Big(\varphi(\hat{\xi}, \hat{u}) + (\gamma(\hat{\xi}) - 1)u_0^{(k)} + K_1(\sigma + W sign(\sigma)\Big)$$

This control law can be implemented to the actual system *Eq (3.1)* by first passing through a chain of $'k'$ integrators which results in a continuous control input to the actual system. This is a clear benefit in terms of chattering attenuation.

Remark 3.1. The coefficients of the conventional sliding surface are chosen by tacking into the dynamic response of the system. However, in real applications, these constants can also be optimized using LMIs methods.

Remark 3.2. The proposed methodology needs the availability of the system output and of its derivatives for the controller implementation. In case the output derivatives are not available for measurements, one can use for instance a finite time sliding mode differentiator like the one proposed in (Levant 2003) to reconstruct them.

Theorem 3.3. *Consider the nonlinear system Eq (3.5) subject to Assumption 1, and 2. If the sliding surface is chosen according to Eq (3.11), the control law* $u^{(k)}$ *is selected according to Eq (3.7) (with control components defined in Eq (3.10) and Eq (3.16)) and the integral term is taken according to Eq (3.13), then the asymptotic convergence condition is satisfied.*

Proof: Consider a Lyapunov function candidate as follows:

$$V = 1/2(\sigma)^2 \qquad \text{Eq (3.17)}$$

The time derivative of *Eq (3.17)* along equation *Eq (3.5)*, yields

$$\dot{V} = \sigma(\sum_{i=1}^{n-1} c_i \xi_{i+1} + \chi(\hat{\xi}, \hat{u}, u^{(k)}) + \zeta^*(\hat{\xi}, t) + u_0^{(k)} + u_1^{(k)} + \dot{z}) \qquad \text{Eq (3.18)}$$

The use of the dynamics of integral term in *Eq (3.13)* reduces *Eq (3.18)* in the following form

$$\dot{V} = \sigma(\chi(\hat{\xi}, \hat{u}, u^{(k)}) + \zeta^*(\hat{\xi}, t) + u_1^{(k)}) \qquad \text{Eq (3.19)}$$

Using Assumption 1 and *Eq (3.16)* in *Eq (3.19)*, one has

$$\dot{V} \leq -\sigma\left(K_1(\sigma + W sign(\sigma))\right)$$

This expression shows that the time derivative of the Lyapunov function is negative which confirms that $\sigma = 0$ is guaranteed in the presence of uncertainties. To show that the system in *Eq (3.5)* is governed by the continuous control component in *Eq (3.10)*, when sliding mode is established, we develop the following corollary.

Corollary 3.4. *The dynamics of the system Eq (3.5), with control law Eq (3.7) and sliding manifold* $\sigma(\hat{\xi}) = 0$*, with* $\sigma(\hat{\xi})$ *defined in Eq (3.11), in sliding mode, is governed by the continuous control component Eq (3.10).*

Proof: To proceed to the proof, differentiating *Eq (3.11)* along the dynamics of the system *Eq (3.5)*, one has

$$\dot{\sigma}(\hat{\xi}) = \sum_{i=1}^{n-1} c_i \xi_{i+1} + \varphi(\hat{\xi}, \hat{u}) + (\gamma(\hat{\xi}) - 1)u^{(k)} + \zeta^*(\hat{\xi}, t) + u^{(k)} + \dot{z}$$

Now, substituting *Eq (3.7)*, one has

$$\dot{\sigma}(\hat{\xi}) = \sum_{i=1}^{n-1} c_i \xi_{i+1} + \varphi(\hat{\xi}, \hat{u}) + (\gamma(\hat{\xi}) - 1)u^{(k)} + \zeta^*(\hat{\xi}, t) + u_0^{(k)} + u_1^{(k)} + \dot{z} \quad Eq\ (3.20)$$

Substituting *Eq (3.13)* into *Eq (3.20)*, one has

$$\dot{\sigma}(\hat{\xi}) = \varphi(\hat{\xi}, \hat{u}) + \gamma(\hat{\xi})u^{(k)} + \zeta^*(\hat{\xi}, t) - u_0^{(k)} \quad Eq\ (3.21)$$

Now, posing $\dot{\sigma}(\hat{\xi}) = 0$ and solving with respect to the control variable $u^{(k)}$, one obtains the so-called equivalent control [3] as

$$u_{eq}^{(k)} = -\frac{1}{\gamma(\hat{\xi})} \left(\varphi(\hat{\xi}, \hat{u}) - u_0^{(k)} + \zeta^*(\hat{\xi}, t) \right) \quad Eq\ (3.22)$$

Now, using *Eq (3.22)* in *Eq (3.5)*, one has

$$\dot{\hat{\xi}}_s = A\hat{\xi}_s + Bu_0^{(k)} \quad Eq\ (3.23)$$

where $\hat{\xi}_s$ is the state of system *Eq (3.5)* in sliding mode. Thus, it is proved that the system in sliding mode operates under the continuous control law and the eigenvalues of the controlled transformed system in sliding mode are those of $A - BK_0^T$.

Note that, the above designed algorithm is used in the output regulation of a couple of examples which will be discussed in the following study.

3.3 Numerical Examples

3.3.1 Example 1: Nonlinear System

The forthcoming system is adapted from [15] and the simulation results of the new control law are compared with the standard results of DSMC. The state space description of the nonlinear system has the following form

$$\dot{x_1} = x_2$$

$$\dot{x_2} = x_1^2 + (x_2^2 + 1)u + x_3 \qquad \textit{Eq (3.24)}$$

$$\dot{x_3} = -x_3 + x_2 x_3^2$$

where $[x_1, x_2, x_3]$ is the state vector and $y = x_1$ is the is measurable output of the plant. The LGCCF of this system can be obtained by three time differentiation of the output along the dynamic of this nonlinear plant. Therefore, the third derivative of the out with respect to the system *Eq (3.24)* takes the following form

$$y^{(3)} = 2x_1x_2 + \dot{u}(x_2^2 + 1) + 2x_2u(x_1^2 + (x_2^2 + 1)u + x_3) - x_3 + x_2x_3^2 \qquad \textit{Eq (3.25)}$$

The Definition 2.1 is satisfied and the zero dynamic mentioned in Definition 2.2, for the aforementioned system becomes[15]

$$\dot{u} + 2u = 0$$

These exponentially stable dynamics shows that the system is minimum phase.The system in *Eq (3.25)* can be written the following LGCCF form

$$\dot{\xi}_i = \xi_{i+1}, i = 1, 2$$

$$\dot{\xi}_3 = \varphi(\hat{\xi}, \hat{u}) + \gamma(\hat{\xi})\dot{u}$$

where $y = \xi_1$, $\gamma(\hat{\xi}) = (x_2^2 + 1)$ and

$$\varphi(\hat{\xi}, \hat{u}) = 2x_1x_2 + 2x_2u(x_1^2 + (x_2^2 + 1)u + x_3) - x_3 + x_2x_3^2$$

The transformation being used here are $\hat{\xi} = [\xi_1, \xi_2, \xi_3]^T = [y, \dot{y}, \ddot{y}]^T$. The sliding surface can be defined by

$$\sigma = a_1\xi_1 + a_2\xi_2 + \xi_3 + z$$

The compenstor dynamics carry the following expression

$$\dot{z} = -\dot{u}_0 + \left(-a_1x_2 - a_2\left(x_1^2 + x_2^2u + x_3\right)\right); \qquad Eq\ (3.26)$$

The expression of the dynamic integral controller is given by

$$\dot{u} = -k_1\xi_1 - k_2\xi_2 - k_3\xi_3 - \frac{1}{\gamma(\hat{\xi})}\left[\varphi(\hat{\xi}, \hat{u}) + (\gamma(\hat{\xi}) - 1)u_0^{(k)} + K_1(\sigma + W sign(\sigma)\right]$$

It is already added that here the results of the proposed control scheme is compared with standard results of literature. The output and their derivative are estimated with the use of the SHGO being presented in section 2.4.1. The design of the observer, in which $\tilde{\xi} = [\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3]^T$ is the estimate of $\xi = [\xi_1, \xi_2, \xi_3]^T$ takes the following form [15]

$$\dot{\tilde{\xi}}_1 = \tilde{\xi}_2 + \left(\frac{\alpha_1}{\epsilon}\right)(\xi_1 - \tilde{\xi}_1)$$

$$\dot{\tilde{\xi}}_2 = \tilde{\xi}_3 + \left(\frac{\alpha_2}{\epsilon^2}\right)(\xi_1 - \tilde{\xi}_1)$$

$$\dot{\tilde{\xi}}_3 = \varphi(\tilde{\xi}, \hat{u}) + \left(\frac{\alpha_3}{\epsilon^3}\right)(\xi_1 - \tilde{\xi}_1)$$

where $[\alpha_1, \alpha_2, \alpha_3]$ are Hurwitz by definition. The sliding manifold based on this estimator appears in the following form

$$\tilde{\sigma} = a_1\tilde{\xi}_1 + a_2\tilde{\xi}_2 + \tilde{\xi}_3 + \tilde{z}$$

and the final dynamic controller expression with these estimates carries the following form

$$\dot{\tilde{u}} = \tilde{u_0}^{(k)} - \frac{1}{\gamma(\tilde{\hat{\xi}})}\left[\varphi(\tilde{\hat{\xi}}, \tilde{\hat{u}}) + (\gamma(\tilde{\hat{\xi}}) - 1)\tilde{u_0}^{(k)} + K_1(\tilde{\sigma} + W sign(\tilde{\sigma})\right]$$

The objective in this example is to regulate the states of the plant to the origin. The state x_1, x_2 and x_3 regulation via the proposed controller are shown in Figure 3.1, Figure 3.2 and Figure 3.3, respectively. It is quite clear that the state

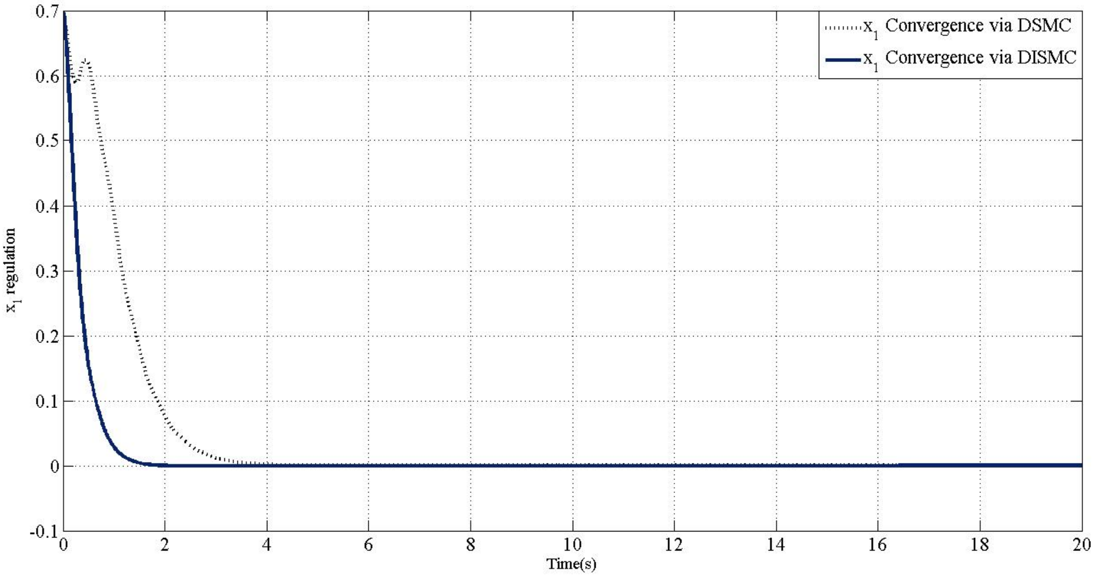


FIGURE 3.1: x_1 regulation via DSMC and DISMC

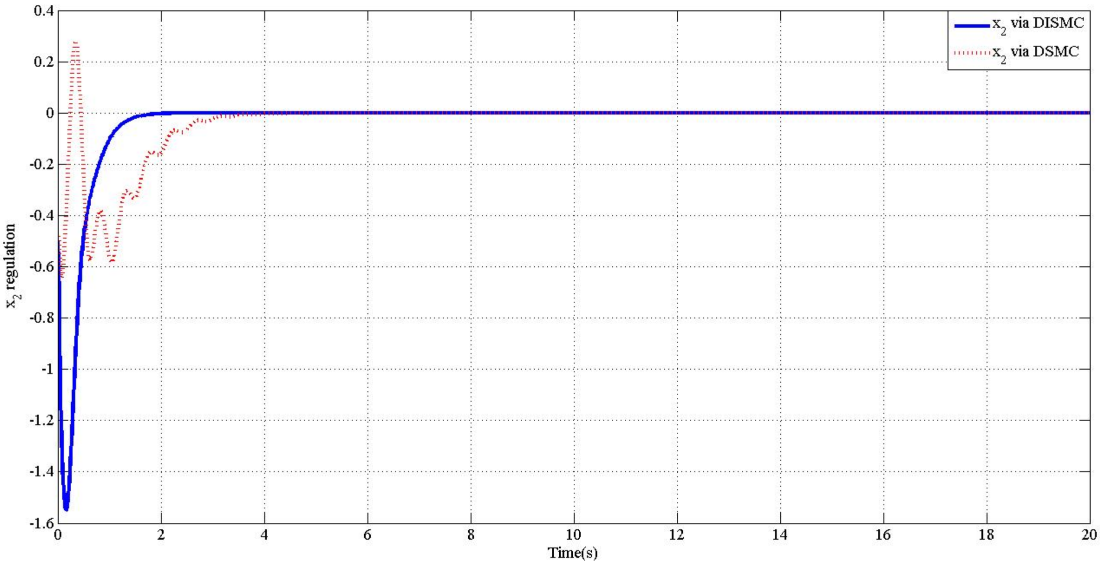


FIGURE 3.2: x_2 regulation via DSMC and DISMC

TABLE 3.1: Values of the controller and differentiator gains used in simulation.

Parameters	k_1	k_2	k_3	K_1	W	a_1	a_2	α_1	α_2	α_3	ϵ
Value	470.22	180.72	11	140	0.005	15	9	5	6	1	0.1

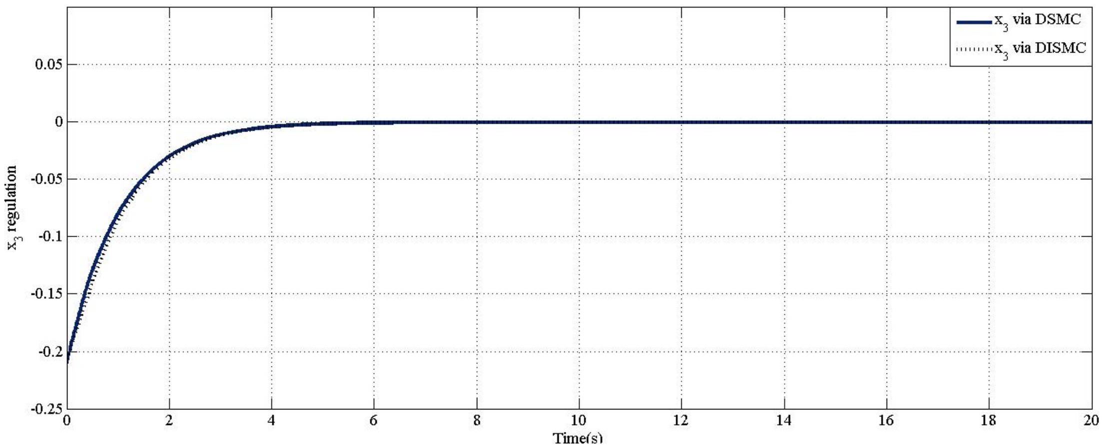


FIGURE 3.3: x_3 regulation via DSMC and DISMC

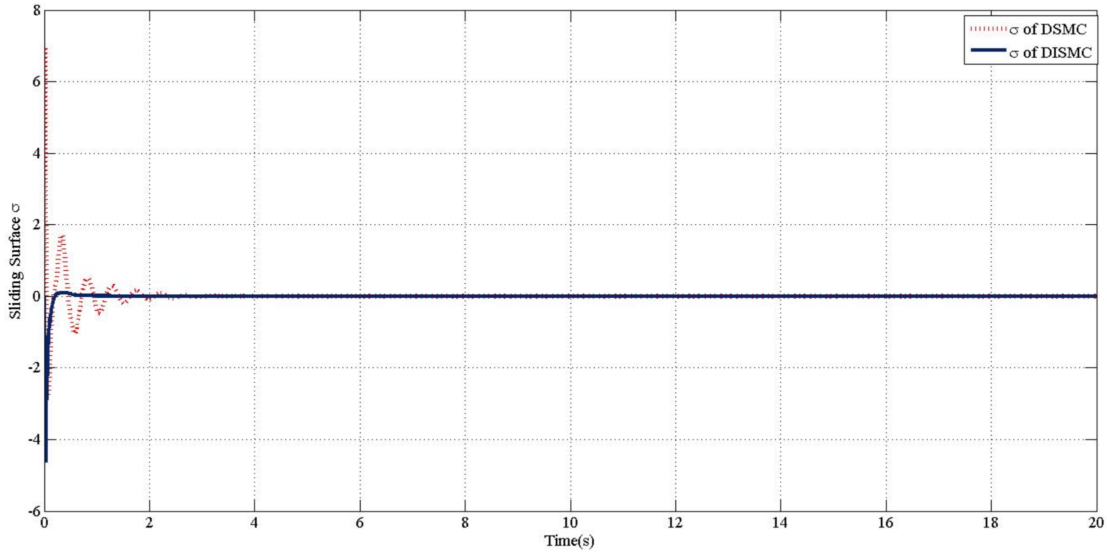


FIGURE 3.4: σ convergence with DSMC and DISMC

convergence via this algorithm is far better than that of [15]. The sliding surface convergence of both the control methodologies in depicted in Figure 3.4. The control efforts of DSMC and the proposed control scheme (DISMC) are illustrated in Figure 3.5. It is obvious that the surface convergence and controller efforts of DISMC is very appealing as compared to the surface convergence and control efforts of [15]. Based on this comparison with standard results, it is decided that DISCM control design scheme is superior to DSMC. The control gains used in this study are give in the Table 3.1.

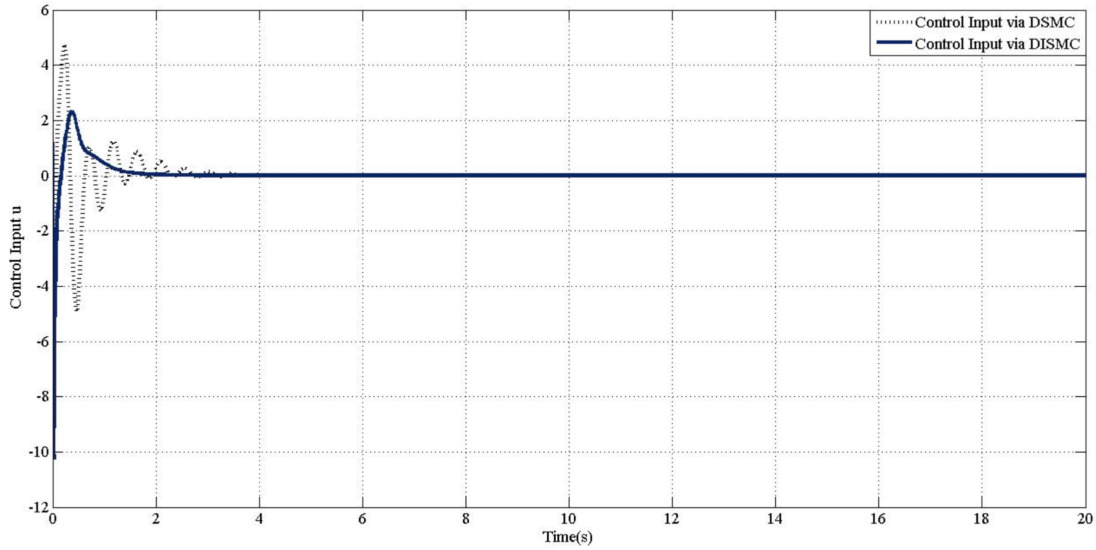


FIGURE 3.5: Control Input u_1 via DSMC and DISMC

3.3.2 Example 2 : Kinematic Car Model

Consider a simple kinematic car model [51] with the following state space representation

$$\dot{x}_1 = wcos(x_3) \qquad \textit{Eq (3.27)}$$

$$\dot{x}_2 = wsin(x_3)$$

$$\dot{x}_3 = (\frac{w}{l})tan(x_4)$$

$$\dot{x}_4 = u + \zeta(x,t)$$

where x_1 and x_2 are the Cartesian coordinates of the rear-axle middle point, x_3 the orientation angle and x_4 the steering angle, u the control input. w is the longitudinal velocity ($w = 10ms^{-1}$, and l is the distance between the two axles ($l = 5m$). The term $\zeta(x,t) = 0.1sin(x_3)^2 + 0.01x_2x_3$ in *Eq (3.27)* represents some unknown bounded uncertainty. The schematic diagram of the car is shown in Figure 3.6. The objective is to regulate the output of the car from some initial position to the equilibrium point (origin). The output of interest is $y = h(x) = x_2$ and relative degree r of the system verses this output function is 3. The LGCCF

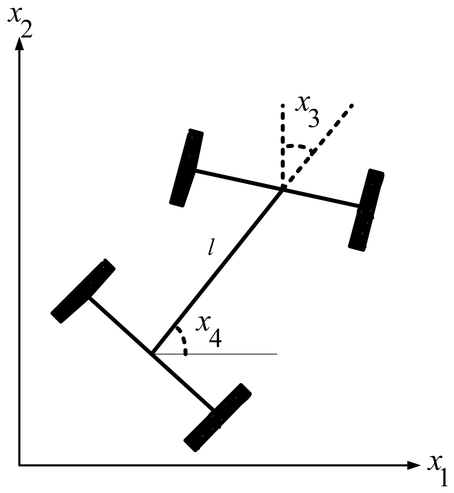


FIGURE 3.6: Kinematic Car Model

form of the system *Eq (3.24)* becomes

$$\dot{\xi}_i = \xi_{i+1}, i = 1, 2, 3$$

$$\dot{\xi}_4 = \varphi(\hat{\xi}, \hat{u}) + \gamma(\hat{\xi})\dot{u} + \zeta^*(\hat{\xi}, \hat{u}, t) \qquad \textit{Eq (3.28)}$$

where $y = \xi_1$, $\hat{\xi} = [\xi_1, \xi_2, \xi_3, \xi_4]^T$

$$\gamma(\hat{\xi}) = cos(x_3)sec^2(x_4) \qquad \textit{Eq (3.29)}$$

and

$$\varphi(\hat{\xi}, \hat{u}) = \frac{w^2}{l}\left(-\frac{w^2}{l^2}cosx_3tan^3x_4 - \left(2\frac{w^2}{l^2} + \frac{w}{l}\right)sinx_3sec^2x_4tanx_4u\right) + \frac{w^2}{l}\left(2cosx_3sec^2x_4tanx_4u\right) \qquad (3.30)$$

The control based on the methodology presented in the previous is given by

$$\dot{u} = -k_1\xi_1 - k_2\xi_2 - k_3\xi_3 - k_4\xi_4 - \frac{1}{\gamma(\hat{\xi})}\left(\varphi(\hat{\xi}, \hat{u}) + (\gamma(\hat{\xi}) - 1)u_0^{(k)} + K_1(\sigma + W sign(\sigma)\right) \qquad (3.31)$$

In the forthcoming section the simulation results are presented to look at the response of the system with the newly developed control law.

Table 3.2: Values of the controller gains used for both DISMC and DSMC controllers.

Parameters	k_1	k_2	k_3	k_4	K_1	W	c_1	c_2	c_3	c_4
Small Gains	31.62	77.32	78.73	34.02	100	0.001	15	30	1	1
High Gains	31.62	77.32	78.73	34.02	650	0.001	30	25	10	1

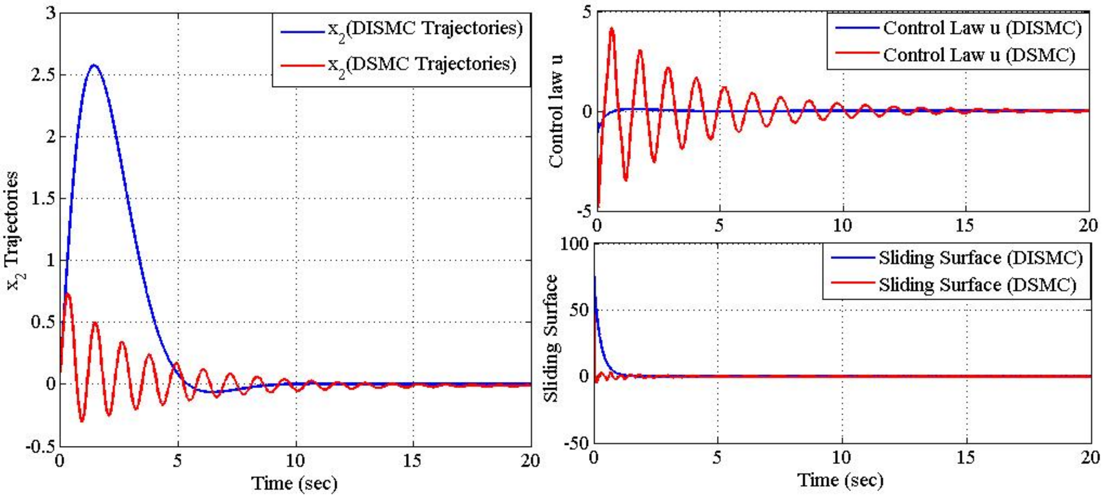


Figure 3.7: DISMC and DSMC Output regulation, control law efforts and sliding surface convergence

3.3.3 Controller Evaluation

The proposed controller is evaluated for the predefined criterion that includes: performance, chattering reduction and robustness. The regulation control of aforementioned academic car model is carried out with DSMC and DISMC and is analyzed in detail with same low control gains and high control gains. The controller gains with small values are defined in Table 3.2. Both DSMC and DISMC are evaluated with same design parameters. The assessment of the controllers is carried out on the basis of states convergence, sliding manifold convergence and controller effort under various types of uncertainties.

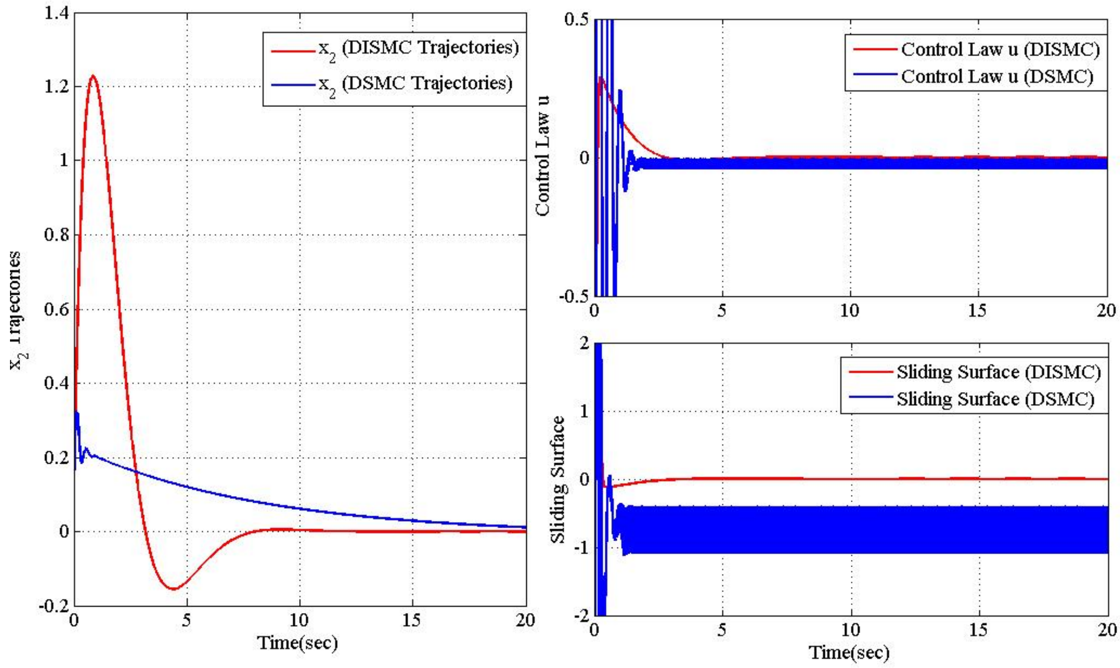


FIGURE 3.8: DISM and DSMC Output regulation, control law efforts and sliding surface convergence

3.3.4 Input Additive Uncertainty

3.3.4.1 Case Study 1: $u = u + \zeta(x,t)$

This uncertainty $\zeta(x,t) = 0.1sinx_3^2 + 0.01x_2x_3$ is of additive nature which is introduced at the input channel which may represent some noise in the input channel. The output convergence, sliding surface convergence and control efforts of DSMC and the proposed DISMC with small control gains are shown in Figure 3.7. It can be observed that with DSMC the state x_2 settles to origin in 8 seconds with oscillatory response and with DISMC x_2 converges to origin in 5 seconds with slight overshoot is the response. Furthermore, a close view of convergence revealed that DSMC instead of converging to origin converged in the vicinity of origin i.e. 0.01 and the proposed controller converged the state exactly to the origin. The convergence of sliding manifold make it clear that DSMC sliding surface converges to -0.25 instead of origin and exhibits chattering in its response. However, DISMC sliding manifold converges exactly to the origin without any chattering. Similarly, the control effort of both the controllers are free of chattering but DSMC exhibits

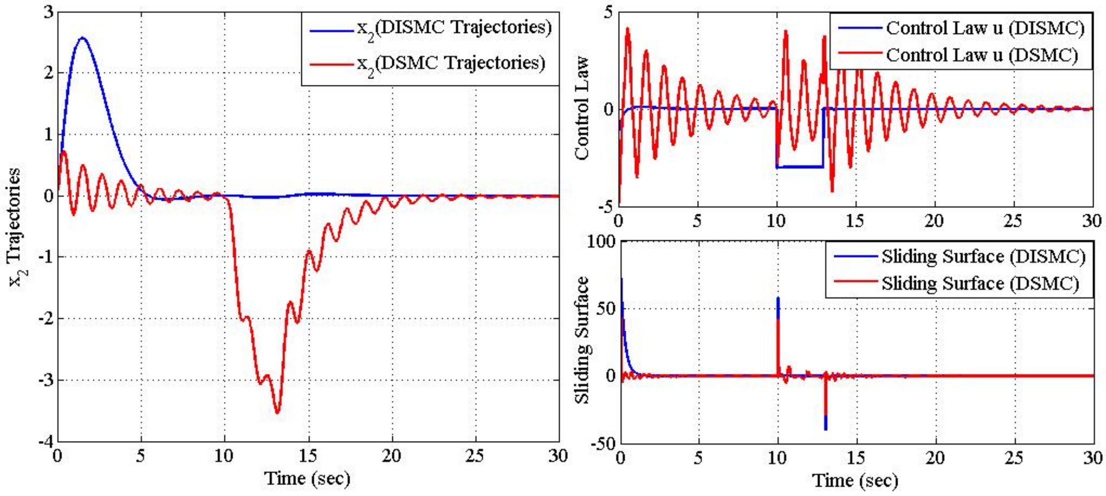


Figure 3.9: DISM and DSMC Output regulation, control law efforts and Sliding surface convergence

oscillations with considerable magnitude. These oscillations may not be too dangerous as chattering phenomena, but still a threat for system/actuator health. In comparison, the proposed controller effort is smooth, free of chattering and oscillations. The comparison of both the control techniques is carried out with high control gains in order to make sure that the proposed control law performs better than DSMC in all case of control gains. This claim is verified in subplots of Figure 3.8. The output convergence under the action of the new control law is better than the DSMC controller. The control efforts of DSMC observe small chattering along with some oscillatory behavior in the very beginning of the process. However, the proposed controller is chatter free. The sliding surface convergence of DISMC is exactly to the origin while the convergence of DSMC is in the vicinity of the origin along with chattering. One more attribute that can be observed is that the proposed controller has to exert much lesser effort as compared to DSMC. Thus, the proposed controller evolves as cheaper controller as compared to DSMC.

3.3.4.2 Case Study 2

In this case, the uncertainty is introduced in the input channel. The control input is incremented by an additive term 3 when $t \in [11, 13]$. This disturbance is independent of the system parameters and is introduced after achieving steady state

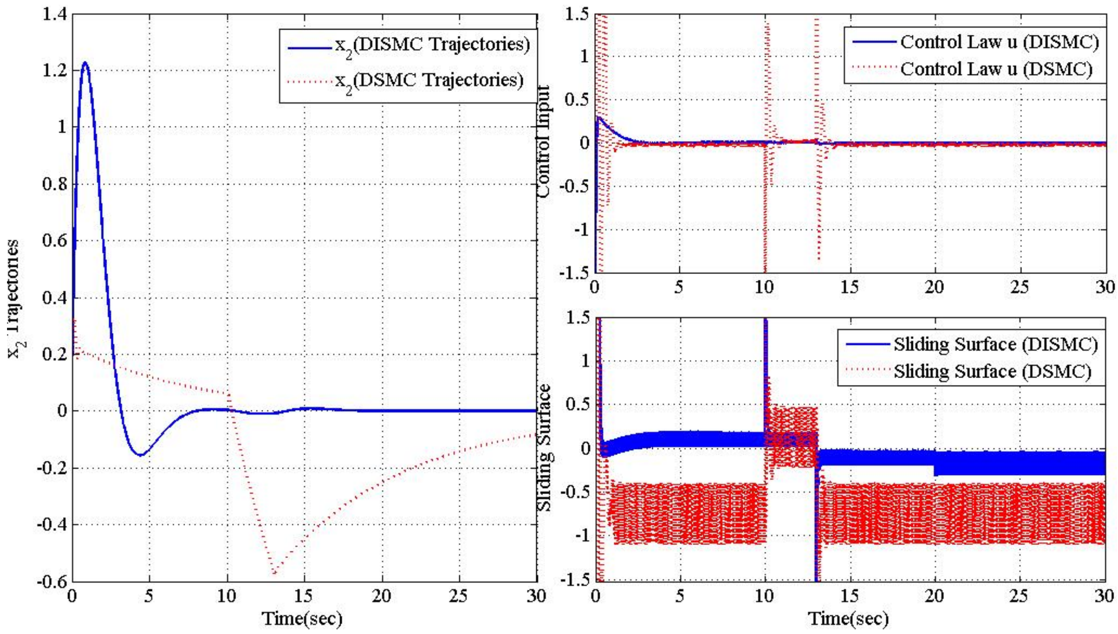


FIGURE 3.10: DISM and DSMC Output regulation, control law efforts and Sliding surface convergence

to evaluate the robustness of the proposed controller. The designed controllers (DSMC and DISMC) with same parameters as in Table 3.2 are chosen for evaluation. The results displayed in Figure 3.9 show that the output trajectory of the system under DSMC deviates from the origin and takes 10 seconds to achieve back the equilibrium position. However, the proposed controller efficiently tackles for the undesirable deviation and the trajectory stay at the origin even in the existence of the disturbance. The respective control efforts show that at the time of the introduction of the disturbance, the switching surface of DSMC oscillates with some undesirable peaks which degrades the robustness and performance of DSM control law. Conversely, the sliding manifold for DISMC remains at '0' with slight peaks, keeping the robustness intact. The proposed controller is tested with high control gains and their results are displayed in Figure 3.10. These results clarify that the proposed controller in output convergence, chattering reduction and performance improvement is better than DSMC.

The controller gains for these results are mentioned in Table 3.2. Table 3.3 contains all the attributes utilized to evaluate the proposed DISMC. Based on the results

Table 3.3: Comparative analysis of proposed controller with Dynamic Sliding mode controller

Attributes	DSMC	DISMC
Robustness	Rejects the disturbance but with deviation from the origin	Effectively rejects the disturbance
Settling Time	8 seconds	5 seconds
Oscillations	Oscillations	No Oscillations
Regulation	To the vicinity of the origin	Exactly to the origin
Overshoot	Exists with oscillations	No Over Shoot
Chattering Analysis	No chattering but oscillatory response	No Chattering
S. Surface Convergence	To the vicinity of origin with Chattering	To origin, No Chattering
Controller Gains	High Gains for desired performance	Small Gains (≈70 % small) for desired performance
Controller Effort	High Controller efforts	Low control efforts
Computational Complexities	Low computation complexities	High computation complexities

enumerated in this table the figures being displayed in the aforementioned case studies, it can be claimed that proposed dynamic integral sliding mode control outshines dynamics sliding mode control in maximum aspects

Remark 3.5. The proposed control methodology provides us a good control in regards to robustness, chattering attenuation and performance improvement but the main problem is that the computation increases. In addition, Like the other output feedback control techniques, some differentiators like semi high gain and Levant differentiators will be needed to implement this control technique on some plant. Furthermore, in practical implementation, the sign function may be replaced by saturation function. Since, the proposed control technique is implemented after passing through a chain of integrators, therefore, a continuous control signal will be used in the control of the plant.

3.4 Summary

A new strategy of control design for single input single output uncertain nonlinear system is presented in this chapter. A Lyapunov base energy function is used to prove the asymptotic stabilization of the plant output in the presence of uncertainties. The results of the illustrative example 1 is compared with standard results of [15]. In example 2, a kinematic car model results is compared for DSMC and DISMC. The proposed control scheme provide chattering free control input with improved performance and enhanced robustness. The simulation are carried out to confirm that this control methodology dominates DSMC control technique in some aspects which are given in the Table 3.3.

In the next chapter, the robustness of the proposed control law is tested in the presence of a class of matched and unmatched uncertainties. The sliding mode is enforced, in finite time, and the stability of the system is theoretically analyzed and demonstrated via a couple of numerical examples

Chapter 4

ROBUST DISMC FOR SISO NONLINEAR SYSTEMS WITH STATES DEPENDENT MATCHED AND UNMATCHED UNCERTAINTIES

Output feedback sliding mode control techniques proved themselves to be the good candidate for systems where only output is measurable and its derivatives can be estimated accurately. Linear systems or systems which could be easily lineralized are addressed in Edwards and Spurgeon [46]. However, nonlinear systems with measurable outputs are for instance dealt with via DSMC ([15], [16], [26] and [27]), where the original system is replaced by LGCC forms. Asymptotic stabilization of LGCC forms by means of DSMC provided satisfactory results. Traditionally, this control methodology based on the SMC theory [3], refers to the case of uncertain systems with matched uncertainties (see, [46] for a definition of this class of uncertainties). However, there are many systems affected by uncertainties which do not satisfy the matching condition. To solve this problem, various methods have been proposed in the literature (see for instance, [52], [53], [54], [55]). All the cited papers relied on a backstepping based SMC design to relax the matching conditions. As already discussed that the robustness of nonlinear systems can be enhanced via the elimination of reaching phase (as proposed in [4]). In this context, Levant et al. [56] used the higher order sliding mode technique (see,e.g., [36], [32], [57], [58], [59] and [60]) in combination of an integral manifold to improve the robustness and alleviate chattering. Choi [61], proposed a linear matrix inequality (LMI)-based sliding surface design method for integral sliding-mode control of systems with unmatched norm bounded uncertainties. Further, Park et al. [62] extended Chois method and proposed a dynamical output feedback variable structure control law with high gain to deal with the same problem. Xiang et al. [63] applied an iterative LMI method to avoid the high gain related problems. In this context, Da Silva et al. [64] developed an algorithm in which the existence and the reachability problems have been formulated using a polytopic description in order to tackle unmatched uncertainties with reduced chattering. Cao and Xu [65] proposed a nonlinear integral-type sliding surface for the system in the presence of both matched and unmatched uncertainties. The stability of the controlled system with unmatched uncertainties depends on the controlled nominal system and on the nature and size of the equivalent unmatched uncertainties. In the aforementioned approaches, robustness is ensured but with a compromise on chattering alleviation. Castanos et al. [66] analyzed the robust features of the integral sliding mode and used theory to overtake the undesirable effects of the uncertainties. Rubagotti et al. [67] extends the work presented in (Castanos et al. [66]) providing a control law which minimizes the effect of the uncertainties. Chang [68], proposed a dynamic output feedback controller design according to an integral sliding mode approach yet, in case of linear systems. Note that in the aforementioned papers it is assumed that all the states of the system are available since they are explicitly used to construct the control law.

In this chapter, an output feedback dynamic sliding mode controller presented in Chapter 3 is employed to a class of SISO nonlinear systems with both matched and unmatched state dependent uncertainties (see, [19] and [20]). The uncertain system output trajectories are asymptotically regulated to zero despite the presence of the uncertainties. This is attained by enforcing, in finite time, a sliding mode along an integral manifold. The use of the integral sliding manifold allows one to subdivide the control design procedure into two steps. First a linear control component is designed by pole placement and then a discontinuous control component is added so as to cope with the uncertainty presence. The design procedure is performed relying on a suitably transformed system which turns out to be in a perturbed LGCC form, in the sense that its form is a LGCC form but affected by uncertainties. As a consequence, the control acting on the original system is obtained as the output of a chain of integrators and is, accordingly, continuous, thus attaining the aim of chattering attenuation. This can be a clear benefit in many applications such as those of mechanical nature, where a discontinuous control action could be inappropriate. Note that the output feedback control of nonlinear systems, which can be put in LGCC form was previously faced, in a preliminary version, in Khan et al. [19]. The rest of the chapter is organized as follows. In Section 4.1, the problem formulation is presented and in Section 4.2 the design of the proposed control law is outlined. In Section 4.3, the stability analysis in the presence of matched and unmatched uncertainties is carried out. The above uncertainties compensation is demonstrated in a couple of numerical examples. This chapter is summarized in the last section.

4.1 Problem Formulations

Consider a nonlinear SISO dynamic system represented by the state equation analogous to that considered in [65]

$$\dot{x} = f(x,t) + g(x,t)\left[u(1+\delta_m) + \Delta g_m(x,t)\right] + f_u(x,t) \qquad \textit{Eq (4.1)}$$

$$y = h(x) \qquad \textit{Eq (4.2)}$$

where $x \in R^n$ is the state vector, $u \in R$ is scalar control input, $f(x,t)$ and $g(x,t)$ are smooth vector fields, δ_m and $\Delta g_m(x,t)$ are matched uncertainties, $f_u(x,t)$ is the unmatch uncertainty vector and $y = h(x)$ is a sufficiently smooth output function. The following assumption is introduced:

Assumption 3. *δ_m,$\Delta g_m(x,t)$ and $f_u(x,t)$ are continuous and bounded with continuous bounded time derivatives $\forall (x,t) \in R^n \times R_+$. i.e., $|\Delta g_m(x,t)| \leq \rho_m$, $|\delta_m| \leq 1 - \epsilon_m$ where ϵ_m is some positive constant, and $||f_u(x,t)|| \leq \rho_u$.*

As in previous chapter, the problem which we want to solve (Problem 1) is that of steering the output to zero asymptotically, i.e. an output regulation problem is considered in the presence of matched and unmatched uncertainties. In order to reformulate the problem, the system *Eq (4.1)* is transformed to the following form

$$y^{(r)} = L_f^r h(x) + L_g(L_f^{(r-1)} h(x))u + \zeta(x,t) \qquad \textit{Eq (4.3)}$$

where $\zeta(x,t)$ represents the matched and unmatched uncertainties collection subject to the conditions and cases proposed in the Chapter 3. The k derivatives of *Eq (4.3)* becomes

$$\begin{aligned} y^{(n)} = {} & L_f^n h(x) + L_g(L_f^{(n-1)} h(x))u + \cdots + L_f L_g L_f^{(r-1)} h(x) u^{(k-1)} \\ & + L_g^2 L_f^{(r-1)} h(x) u u^{(k-1)} + L_g L_f^{(r-1)} h(x) u^{(k)} + \zeta^*(x,u,\dot{u},\ldots,u^{(n-1)},t) \end{aligned} \qquad (4.4)$$

where $\zeta^*(x,u,\dot{u},\ldots,u^{(n-1)},t)$ is the lumped uncertainty which represents the collection of matched and unmatched uncertainties with their time derivatives.

The system *Eq (4.4)* becomes

$$y^{(n)} = \varphi(\hat{y}, \hat{u}) + \gamma(\hat{y})u^{(k)} + \zeta^*(x, u, \dot{u}, \ldots, u^{(n-1)}, t) \tag{4.5}$$

$$= \varphi(\hat{y}, \hat{u}) + \gamma(\hat{\xi}) \left[u^{(k)}(1 + \delta_m) + \Delta G_m(\hat{\xi}, \hat{u}, t)\right] + F_u(\hat{\xi}, \hat{u}, t)$$

where

$$\varphi(\hat{y}, \hat{u}) = L_f^n h(x) + L_g(L_f^{(n-1)} h(x))u + \cdots + L_f L_g L_f^{(r-1)} h(x) u^{(k-1)} + L_g^2 L_f^{(r-1)} h(x) u u^{(k-1)})$$

and

$$\gamma(\hat{y}) = L_g L_f^{(r-1)} h(x)$$

where $\hat{y} = (y, \dot{y}, \cdots, y^{(n-1)})$, $\hat{u} = (u, \dot{u}, \cdots, u^{(n-1)})$, δ_m, $\Delta G_m(\hat{\xi}, \hat{u}, t)$ are the matched uncertainty terms and $F_u(\hat{\xi}, \hat{u}, t)$ is the unmatched uncertainty term. Now, once again using the transformation $y^{(i-1)} = \xi_i$ [27], $\hat{y} = \hat{\xi}$ and $\zeta_1(\hat{\xi}, \hat{u}, t) = L_{f_u}^i h(x)$ for $i = 1, 2, \cdots, n$, the system *Eq (4.4)* can be written as as follows

$$\dot{\xi}_1 = \xi_2 + \zeta_1(\hat{\xi}, \hat{u}, t)$$

$$\dot{\xi}_2 = \xi_3 + \zeta_2(\hat{\xi}, \hat{u}, t)$$

$$\vdots$$

$$\dot{\xi}_n = \varphi(\hat{\xi}, \hat{u}) + \gamma(\hat{\xi}) \left[u^{(k)}(1 + \delta_m) + \Delta G_m(\hat{\xi}, \hat{u}, t)\right] + F_u(\hat{\xi}, \hat{u}, t) \qquad \textit{Eq (4.6)}$$

The representation in *Eq (4.6)* is analogous to the so-called Local Generalized Controllable Canonical (LGCC) form or Fliess Controllable Canonical (Fliess 1990), in the sense that it differs from the basic LGCC form since it is also affected by uncertainties. With reference to system *Eq (4.6)*, the following assumption (which is an alternative form of Assumption 3 is introduced:

Assumption 4. *Assume that* $|\varphi(\hat{\xi}, \hat{u}, t)| \leq C$, $|\gamma(\hat{\xi})| \leq K_M$, $|\Delta G_m(\hat{\xi}, \hat{u}, t)| \leq \lambda_1$, $|F_u(\hat{\xi}, \hat{u}, t)| \leq \lambda_2$ *and* $|\zeta_i(\hat{\xi}, \hat{u}, t)| \leq \mu_i$ $i = 1, 2, , n-1$, *where* λ_1, λ_2 *and* μ_i *are positive constants. Furthermore, consider that* $\zeta_1(\hat{\xi}, \hat{u}, t) + \zeta_2(\hat{\xi}, \hat{u}, t) + \cdots + F_u(\hat{\xi}, \hat{u}, t) \cong \Delta\phi(\hat{\xi}, \hat{u}, t)$ *and* $|\Delta\phi(\hat{\xi}, \hat{u}, t)| \leq \tau$.

The nominal system corresponding to *Eq (4.6)* can be obtained by putting $\zeta_i(\hat{\xi}, \hat{u}, t) = 0$ and $\Delta G_m(\hat{\xi}, \hat{u}, t) = 0$, $\delta_m = 0$ and $F_u(\hat{\xi}, \hat{u}, t) = 0$.

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_n &= \varphi(\hat{\xi}, \hat{u}) + \gamma(\hat{\xi})u^{(k)} \\
&= \Phi(\hat{\xi}, \hat{u}, u^{(k)})
\end{aligned} \qquad Eq\ (4.7)$$

Note that the nominal system referred in *Eq (4.7)* is similar to that reported in *Eq (3.6)* and satisfy the Assumption 2.
Now the original control problem (Problem 1) can be reformulated with reference to system *Eq (4.6)* under Assumption 4, and to the nominal system in *Eq (4.7)* subject to Assumption 2. The new problem (Problem 2) is that of steering the state vector $\xi = [\xi_1, \xi_2, ..., \xi_n]$ of system *Eq (4.6)* to zero asymptotically in spite of the presence of matched and unmatched uncertainties, i.e. a robust state regulation problem is now considered. Clearly the solution to Problem 2 implies the solution to Problem 1, since $\xi_1 = y = h(x)$.

4.2 Controller Design

In analogy with Chapter 3, where only the presence of matched uncertainties was considered, we propose a control law of dynamic nature which can be expressed as reported in *Eq (3.7)*. The first part $u_0^{(k)} \in R$ is continuous and stabilizes the system at the equilibrium point, while the second part $u_1^{(k)} \in R$ is discontinuous in nature and can be classified as an integral SMC. Its role is to reject uncertainties. In the next subsections, the design of $u_0^{(k)} \in R$ and $u_1^{(k)} \in R$ is outlined. Starting from the nominal case and then moving to the case in which the presence of matched and unmatched uncertainties is also considered.

4.2.1 Design of $u_0^{(k)}$

The problem formulation for the design of $u_0^{(k)}$ is similar to that presented in the Section 3.2.1. The only difference which is used here is that this component is designed via pole placement (for the sake of simplicity). Therefore, one has

$$u_0^{(k)} = -K_0^T \xi \qquad \textit{Eq (4.8)}$$

4.2.2 Design of $u_1^{(k)}$

This component of the control law is designed in similar fashion to that presented in Section 3.2.2 with only difference in the reachability condition. The reachability used there is strong reachability while here we use the most famous reachability condition [3] as follows

$$\dot{\sigma}(\xi) = -K sign(\sigma) \qquad \textit{Eq (4.9)}$$

Thus the controller expression reported in *Eq (3.16)* appears as follows

$$u_1^{(k)} = -\frac{1}{\gamma(\hat{\xi})}\left(\varphi(\hat{\xi}, \hat{u}) + (\gamma(\hat{\xi}) - 1)u_0^{(k)} + K sign(\sigma)\right) \qquad \textit{Eq (4.10)}$$

The final expression of the control becomes

$$u^{(k)} = u_0^{(k)} - \frac{1}{\gamma(\hat{\xi})}\left(\varphi(\hat{\xi}, \hat{u}) + (\gamma(\hat{\xi}) - 1)u_0^{(k)} + K sign(\sigma)\right) \qquad \textit{Eq (4.11)}$$

Note that, like in previous case of Chapter 3, this control law can be implemented by integrating the derivative of the control, $u^{(k)}$, k times so that the control input actually applied to the system is continuous. This can be a benefit for various class of systems such as those of mechanical type, for which a discontinuous control action could be disruptive.

4.3 Stability Analysis in the Presence of uncertainties

In this section, the proposed control law when applied to the uncertain nonlinear system in question is theoretically analyzed. First the case in which only matched uncertainties are present will be discussed, and then, the more general case of both matched and unmatched uncertainties will be considered.

4.3.1 The System Operating Under Matched Uncertainties

Now we assume that the system operates only under matched uncertainties. Thus, system *Eq (4.6)* with matched uncertainties becomes

$$\dot{\xi}_1 = \xi_2$$

$$\dot{\xi}_2 = \xi_3$$

$$\vdots$$

$$\dot{\xi}_n = \varphi(\hat{\xi}, \hat{u}) + \gamma(\hat{\xi}) \left[u^{(k)}(1 + \delta_m) + \Delta G_m(\hat{\xi}, \hat{u}, t) \right] \qquad \textit{Eq (4.12)}$$

To show that this system is stabilized in finite time in the presence of matched uncertainties, the following theorem can be stated.

Theorem 4.1. *Consider that Assumptions 2 and 4 are satisfied. The sliding surface is chosen as $\sigma(\xi) = 0$, where $\sigma(\xi)$ is defined in Eq (3.11), and the control law $u^{(k)}$ is selected according to Eq (4.11). If the gain is chosen according to the following condition*

$$K \geq \frac{1}{(2 - \epsilon_m)} [(1 - \epsilon_m)(||u_0^{(k)}|| + C) + K_M \beta_1 + \eta_1] \qquad \textit{Eq (4.13)}$$

where η_1 is some positive constant, then, the finite time enforcement of a sliding mode on $\sigma(\xi) = 0$ is guaranteed in the presence of matched uncertainties.

Proof: To prove that the sliding mode can be enforced in finite time, differentiating *Eq (3.11)* along the dynamics of *Eq (4.12)*, and then substituting *Eq (4.11)*, one has

$$\dot{\sigma}(\xi) = \sum_{i=1}^{n-1} c_i \xi_{i+1} + \varphi(\hat{\xi}, \hat{u}) + \gamma(\hat{\xi}) \left[u^{(k)}(1 + \delta_m) + \Delta G_m(\hat{\xi}, \hat{u}, t) \right] + \dot{z} \quad Eq\ (4.14)$$

simplifying, one has

$$\dot{\sigma}(\xi) = \sum_{i=1}^{n-1} c_i \xi_{i+1} + \varphi(\hat{\xi}, \hat{u}) + \gamma(\hat{\xi}) u^{(k)} + \gamma(\hat{\xi}) u^{(k)} \delta_m + \gamma(\hat{\xi}) \Delta G_m(\hat{\xi}, \hat{u}, t) + \dot{z} \quad Eq\ (4.15)$$

Now, inserting the expression of the integral term from *Eq (3.13)*, the above expression reduces to the following form

$$\dot{\sigma}(\xi) = \varphi(\hat{\xi}, \hat{u}) + \left(\gamma(\hat{\xi}) - 1 \right) u_0^{(k)} + \gamma(\hat{\xi}) u_1^{(k)} + \gamma(\hat{\xi}) u^{(k)} \delta_m + \gamma(\hat{\xi}) \Delta G_m(\hat{\xi}, \hat{u}, t) \quad Eq\ (4.16)$$

Using the expression of the discontinuous controller *Eq (4.10)* in *Eq (4.16)*, one has

$$\dot{\sigma}(\xi) = -K sign(\sigma) + \delta_m [-\varphi(\hat{\xi}, \hat{u}, t) + u_0^{(k)} - K sign(\sigma)] + \gamma(\hat{\xi}) \Delta G_m(\hat{\xi}, \hat{u}, t) \quad (4.17)$$

Now, by considering as a Lyapunov candidate function $V = \frac{1}{2}\sigma^2$, the time derivative of this function along *Eq (4.17)* becomes

$$\dot{V} = \sigma[-K sign(\sigma) + \delta_m [-\varphi(\hat{\xi}, \hat{u}, t) + u_0^{(k)} - K sign(\sigma)] + \gamma(\hat{\xi}) \Delta G_m(\hat{\xi}, \hat{u}, t)] \quad (4.18)$$

or

$$\dot{V} \leq |\sigma| [-K(1 + |\delta_m|) + |\delta_m| |\varphi(\hat{\xi}, \hat{u})| + |\delta_m| |u_0^{(k)}| + |\gamma(\hat{\xi})| |\Delta G_m(\hat{\xi}, \hat{u}, t)|] \quad (4.19)$$

In view of Assumption 4, the above expression can be written as

$$\dot{V} \leq |\sigma| [-(2 - \epsilon_m) K + (1 - \epsilon_m) |u_0^{(k)}| + (1 - \epsilon_m) C + K_M \beta_1] \quad (4.20)$$

$$\begin{aligned} \dot{V} &\leq -\eta_1 |\sigma| \\ &< 0 \end{aligned} \quad (4.21)$$

Provided that

$$K \geq \frac{1}{(2-\epsilon_m)}[(1-\epsilon_m)(|u_0^{(k)}| + C) + K_M\beta_1 + \eta_1] \qquad \textit{Eq (4.22)}$$

as in *Eq (4.13)*. Note that *Eq (4.21)* can also be written as

$$\dot{V} + \sqrt{2}\eta_1|\sigma| < 0 \qquad (4.23)$$

This implies that $\sigma(\xi)$ converges to zero in a finite time t_s (see Edwards and Spurgeon [46]), such that

$$t_s \leq \sqrt{2}\eta_1^{-1}\sqrt{V}(\sigma(0)) \qquad \textit{Eq (4.24)}$$

which completes the proof.

Remark 4.2. The Corollary 4.5 remains true for the aforementioned study of matched uncertainties and the expression of the $u_{eq}^{(k)}$ appears as follows

$$u_{eq}^{(k)} = -\frac{1}{\gamma(\hat{\xi})(1+\delta_m)}\left(\varphi(\hat{\xi},\hat{u}) - u_0^{(k)} + \gamma(\hat{\xi})\Delta G_m(\hat{\xi},\hat{u},t)\right) \qquad \textit{Eq (4.25)}$$

4.3.2 The System Operating Under both Matched and Unmatched Uncertainties

In this subsection, it is now assumed that the considered system operates under both matched and unmatched uncertainties and the control objective is to regulate the output of the system in the presence of these uncertainties. To prove that the proposed control law is capable of compensating for these uncertain terms, the following theorem can be stated.

Theorem 4.3. *Consider that Assumptions 2 and 4 are satisfied. The sliding surface is chosen as $\sigma(\xi) = 0$, where $\sigma(\xi)$ is defined in Eq (3.11), and the control law $u^{(k)}$ is selected according to Eq (4.11). If the gain is chosen according to the following condition*

$$K \geq \frac{1}{(2-\epsilon_m)}[(1-\epsilon_m)(||u_0^{(k)}|| + C) + K_M\beta_1 + \tau + \eta_2] \qquad \textit{Eq (4.26)}$$

where η_2 is some positive constant, then, the finite time enforcement of a sliding mode on $\sigma(\xi) = 0$ is guaranteed in the presence of matched and unmatched uncertainties.

Proof: Consider the time derivatives of *Eq (3.11)* along the dynamics of *Eq (4.6)*, and then substituting *Eq (4.11)*, one has

$$\dot{\sigma}(\xi) = \sum_{i=1}^{n-1} c_i \xi_{i+1} + \varphi(\hat{\xi}, \hat{u}) + \gamma(\hat{\xi}) \left[u^{(k)}(1 + \delta_m) + \Delta G_m(\hat{\xi}, \hat{u}, t) \right] + F_u(\hat{\xi}, \hat{u}, t) + \dot{z} \quad \textit{Eq (4.27)}$$

simplifying, one has

$$\dot{\sigma}(\xi) = \sum_{i=1}^{n-1} c_i \xi_{i+1} + \varphi(\hat{\xi}, \hat{u}) + \gamma(\hat{\xi}) u^{(k)} + \gamma(\hat{\xi}) u^{(k)} \delta_m + \gamma(\hat{\xi}) \Delta G_m(\hat{\xi}, \hat{u}, t) + F_u(\hat{\xi}, \hat{u}, t) + \dot{z} \quad \textit{Eq (4.28)}$$

Now, inserting the expression of the integral term form *Eq (3.13)*, the above expression reduces to the following form

$$\dot{\sigma}(\xi) = \varphi(\hat{\xi}, \hat{u}) + \left(\gamma(\hat{\xi}) - 1 \right) u_0^{(k)} + \gamma(\hat{\xi}) u_1^{(k)} + \gamma(\hat{\xi}) u^{(k)} \delta_m + \gamma(\hat{\xi}) \Delta G_m(\hat{\xi}, \hat{u}, t) + F_u(\hat{\xi}, \hat{u}, t) \quad \textit{Eq (4.29)}$$

Using the expression of the discontinuous controller *Eq (4.10)* in *Eq (4.29)*, one has

$$\begin{aligned} \dot{\sigma}(\xi) = & -K sign(\sigma) + \delta_m[-\varphi(\hat{\xi}, \hat{u}, t) + u_0^{(k)} - K sign(\sigma)] \\ & + \gamma(\hat{\xi}) \Delta G_m(\hat{\xi}, \hat{u}, t) + F_u(\hat{\xi}, \hat{u}, t) \end{aligned} \tag{4.30}$$

Now, by considering as a Lyapunov candidate function $V = \frac{1}{2}\sigma^2$, the time derivative of this function along *Eq (4.30)* becomes

$$\begin{aligned} \dot{V} = \sigma[-K sign(\sigma) + \delta_m[-\varphi(\hat{\xi}, \hat{u}, t) + u_0^{(k)} - K sign(\sigma)] + \gamma(\hat{\xi}) \Delta G_m(\hat{\xi}, \hat{u}, t) \\ + F_u(\hat{\xi}, \hat{u}, t)] \end{aligned} \tag{4.31}$$

or

$$\dot{V} \leq |\sigma|[-K(1 + |\delta_m|) + |\delta_m||\varphi(\hat{\xi}, \hat{u})| + |\delta_m||u_0^{(k)}| + |\gamma(\hat{\xi})||\Delta G_m(\hat{\xi}, \hat{u}, t)| + |F_u(\hat{\xi}, \hat{u}, t)|] \tag{4.32}$$

In view of Assumption 4, the above expression can be written as

$$\dot{V} \leq |\sigma|[-(2-\epsilon_m)K + (1-\epsilon_m)|u_0^{(k)}| + (1-\epsilon_m)C + K_M\beta_1 + \tau] \quad (4.33)$$

$$\begin{aligned}\dot{V} &\leq -\eta_2|\sigma| \\ &< 0\end{aligned} \quad (4.34)$$

Provided that

$$K \geq \frac{1}{(2-\epsilon_m)}[(1-\epsilon_m)(|u_0^{(k)}| + C) + K_M\beta_1 + \tau + \eta_2] \quad \textit{Eq (4.35)}$$

as in *Eq (4.26)*. The expression in *Eq (4.35)* can be placed in the same format like that of *Eq (4.23)*. Note that the finite time in this case is given by the formula in *Eq (4.24)* with η_2 instead of η_1. Thus it is confirmed that, when the gain of the control law *Eq (4.11)* is selected according to *Eq (4.26)*, the finite time enforcement of the sliding mode is guaranteed in the presence of matched and unmatched uncertainties, which proves the theorem.

Corollary 4.4. *The dynamics of system Eq (4.6), with control law Eq (4.11) and an integral manifold $\sigma(\xi) = 0$, with $\sigma(\xi)$ defined in Eq (3.11), in sliding mode is governed by the linear control law Eq (4.8).*

Proof: The proof can be performed by following the same procedure as in the proof of Corollary 1(Chapter 3), with the only difference that in this case the equivalent control is equal to

$$u_{eq}^{(k)} = -\frac{1}{\gamma(\hat{\xi})(1+\delta_m)}[\varphi(\hat{\xi},\hat{u}) - u_0^{(k)} + \gamma(\hat{\xi})\Delta G_m(\hat{\xi},\hat{u},t) + F_u(\hat{\xi},\hat{u},t)] \quad \textit{Eq (4.36)}$$

4.4 Numerical Examples

4.4.1 System with Relative Degree Two

Consider the nonlinear system reported in *Eq (3.24)* with the following form

$$\dot{x_1} = x_2 + f_1(x,t)$$

$$\dot{x_2} = x_1^2 + (x_2^2+1)(u(1+\delta_m) + \Delta_m g_m(x,t)) + x_3 + f_2(x,t) \qquad \textit{Eq (4.37)}$$

$$\dot{x_3} = -x_3 + x_2 x_3^2 + f_3(x,t)$$

The terms δ_m and Δg_m are matched uncertainties and $f_i(x,t)$, for $i = 1,2,3$ are components of the unmatched uncertainty which satisfy Assumptions 1 and 2 and these terms contribute to the system uncertainty with the following mathematical expressions.

$$f_1(x,t) = -x_3 + x_2 x_3^2 + (-x_3 + x_2 x_3^2)^2 + 0.25sin(t)cos(3x_2) + 0.26$$

$$f_2(x,t) = 0.25sin(t)cos(3x_2) + 0.1$$

$$f_3(x,t) = -x_3 + x_2 x_3^2 + 3(-x_3 + x_2 x_3^2)^2 + 0.25sin(t)cos(3x_2) + 0.1$$

$$\Delta g_m(x,t) = 3(-x_3 + x_2 x_3^2)$$

$$\delta_m = 0.3cos(\pi t x_2)$$

Following the procedure of the Section 3.3.1, the control law becomes

$$\dot{u} = -k_1\xi_1 - k_2\xi_2 - k_3\xi_3 - \frac{1}{\gamma(\hat{\xi})}\left[\varphi(\hat{\xi},\hat{u}) + (\gamma(\hat{\xi})-1)u_0^{(k)} + K_1(\sigma + Wsign(\sigma)\right]$$

Eq (4.38)

In this study we compare the results of the proposed control law with that of quasi continuous high order sliding mode controller proposed by Levant in [59]. To apply such an approach, we denote as

$$s = x_1$$

$$\dot{s} = x_2$$

so that the expression of the Quasi Continuous Sliding Mode Controller in case of relative degree (2-QCSMC) takes the following form

$$u = -\frac{\alpha \dot{s} + |s|^{1/2} sign(s)}{|\dot{s}| + |s|^{1/2}} \qquad Eq\ (4.39)$$

where α is the controller gain which can be selected according to [38]. As proved in Levant [59], the control law *Eq (4.39)* provides a finite time sliding mode of the system with a control law which is continuous everywhere except on the second order sliding manifold $s = \dot{s} = 0$.

4.4.1.1 System Operated with Matched Uncertainty

In this study, the system with matched uncertainties (i.e., with $f_i(x,t) = 0$ for $i = 1, 2, 3$) is simulated to confirm the aforementioned claim of the compensation of uncertain terms. This test with matched uncertainty is also performed with 2-QCSMC previously mentioned. The results are reported in Figures 4.1 and 4.2. In these Figures, it can be seen that the output system with state vector $[\xi_1, \xi_2, \xi_3]^T$ is regulated in the presence of uncertainties. It is noticeable that the proposed methodology provides a satisfactory regulation of the system output via a continuous control law. The 2-QCSMC also provides excellent performance yet with a control law which becomes discontinuous when the output regulation objective is attained. Apart from that, both the controllers need to use a differentiator ([32] and [59]) to construct the derivatives of the output variable necessary in the control laws.

4.4.1.2 The System Operated under Matched and Unmatched Uncertainties

In this section, the test with both matched and unmatched uncertainty is performed. The results with the proposed control law are depicted in Figure 4.3 . These simulation results confirm the robust and chattering free nature of the proposed controller as well as its capability of efficiently solving the regulation problem even in this particularly critical case. In view of the nature of the uncertainty now considered, we cannot compare our results with those of the 2-QCSMC algorithm, since that algorithm was designed under the assumption of having only matched uncertainty [59].

4.4.2 System with Relative Degree Three

Consider the nonlinear system being presented in *Eq (3.27)* with some state dependent unmatched uncertainties

$$\dot{x}_1 = wcos(x_3) + f_1(x,t)$$

$$\dot{x}_2 = wsin(x_3) + f_2(x,t)$$

$$\dot{x}_3 = \frac{w}{l} tan(x_4) + f_3(x,t)$$

$$\dot{x}_4 = u(1+\delta_m) + \Delta G_m(x,t) + f_4(x,t)$$

The description of the models parameters along with figure are presented in Example 2 of Chapter 3. The terms δ_m, $\Delta G_m(x,t)$ are matched uncertainties and $f_i(x,t)$ for $i = 1,2,3,4$ are the components of the unmatched uncertainty which satisfy Assumptions 3 and 4 and these terms contribute to the system uncertainty with the following mathematical expressions.

$$f_1(x,t) = 0.3x_2^2 x_3 sin(x_1), f_2(x,t) = 0.2x_2 sin(x_1)$$

$$f_3(x,t) = 0.2x_2 sin(3x_1), f_4(x,t) = 0.1x_2 sin(3x_1)$$

$$\Delta_m(x,t) = 0.4x_2 sin(3x_1), \delta_m = 0.1x_1 sin(3x_1) + 0.1$$

So that $f_u(x,t)$ introduced in *Eq (4.1)* is here $f_u(x,t) = diag(f_i(x,t))$, only $i = 1,2,3,4$ and $||f_u(x,t)|| \leq 0.3||x||$. The objective is to regulate the variable which is regulated as output of the car from some initial condition to the equilibrium point (origin) in the presence of these uncertainties.

In this test, once again we compare the results of the proposed control law with that of the 3-Quasi Continuous Sliding Mode Control in case of relative degree 3 (3-QCSMC) [59]. In this case the sliding manifold is defined as follows

$$s = x_2$$

$$\dot{s} = wsin(x_3)$$

$$\ddot{s} = \frac{w^2}{l} cos(x_3)tan(x_4)$$

The expression of the quasi continuous controllers is give by

$$u = -\alpha \frac{\ddot{s} + 2(|\dot{s}| + |s|^{2/3})^{1/2}(\dot{s} + |s|^{2/3} sign(s))}{|\ddot{s}| + 2(|\dot{s}| + |s|^{2/3})^{1/2}} \qquad Eq\ (4.40)$$

where α is the controller gain which can be selected according to [38]. As proved in Levant [59], the control law *Eq (4.40)* provides a finite time sliding mode of the system with a control law which is continuous everywhere except on the third order sliding manifold $s = \dot{s} = \ddot{s} = 0$.

4.4.2.1 System Operated with Matched Uncertainty

In this study, the system with matched uncertainties (i.e., $f_i(x,t) = 0$ with for $i = 1, 2, 3, 4$) is simulated to confirm the aforementioned claim of the compensation of uncertain terms. This test with matched uncertainty is also performed with 3-QCSMC previously mentioned. The results are reported in Figures 4.4 and 4.5. In these Figures, it can be seen that the output system with state vector $[\xi_1, \xi_2, \xi_3, \xi_4]^T$ is regulated in the presence of uncertainties. It is noticeable that the proposed methodology provides a satisfactory regulation of the system output via a continuous control law. The 3-QCSMC also provides excellent performance yet with a control law which becomes discontinuous when the output regulation objective is attained. Apart from that, again we note that both the controllers need to use a differentiator ([32] and [59]) to construct the derivatives of the output variable necessary in the control laws.

4.4.2.2 System with Matched and Unmatched Uncertainties

In this section, the test with both matched and unmatched uncertainty is performed. The results with the proposed control law are depicted in Figures 4.6. These simulation results confirm the robust and chattering free nature of the proposed controller as well as it capability of efficiently solving the regulation problem even in this particularly critical case. In view of the nature of the uncertainty now

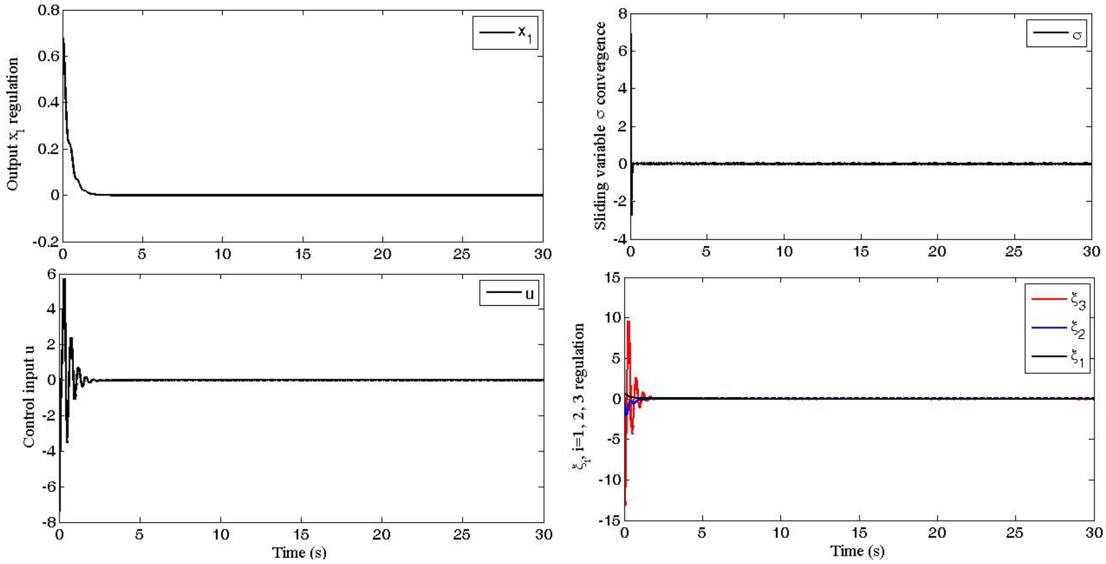


FIGURE 4.1: Output regulation, control effort, sliding variable convergence and $[\xi_1, \xi_2, \xi_3]^T$ regulation in the presence of matched uncertainty via the proposed control law.

TABLE 4.1: Gains of the Controllers

Constants	k_1	k_2	k_3	k_4	c_1	c_2	c_3	K_1 or α
2-QCSMC	–	–	–	–	–	–	–	4
3-QCSMC	–	–	–	–	–	–	–	20
Propose control with $r = 2$	490.2	180.7	5.9	–	6	5	–	230
Propose control with $r = 3$	550	642.5	103.8	22.4	10	30	5	20

considered, we cannot compare our results with those of the 3-QCSMC algorithm, since that algorithm was designed under the assumption of having only matched uncertainty [59].

Note that the controller gains and the controllers parameters in both the tests are listed in following Table 4.1.

4.5 Summary

In this chapter, an output feedback dynamic sliding mode controller is presented capable of dealing with a class of SISO nonlinear systems with both matched and

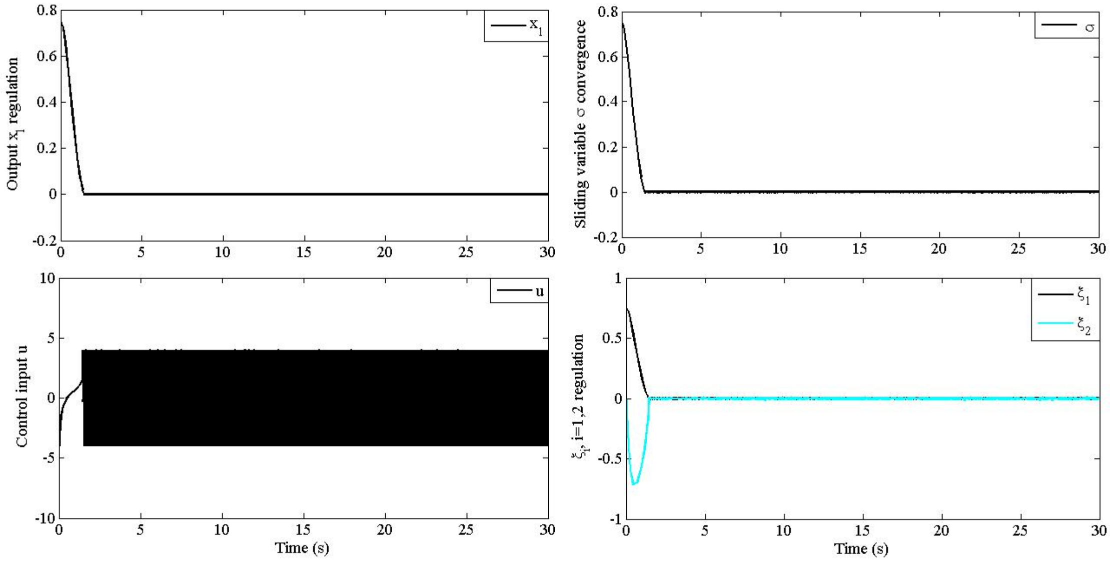


FIGURE 4.2: Output regulation, control effort, sliding variable convergence and $[\xi_1, \xi_2]^T$ regulation in the presence of matched uncertainty via the 2-QCSMC.

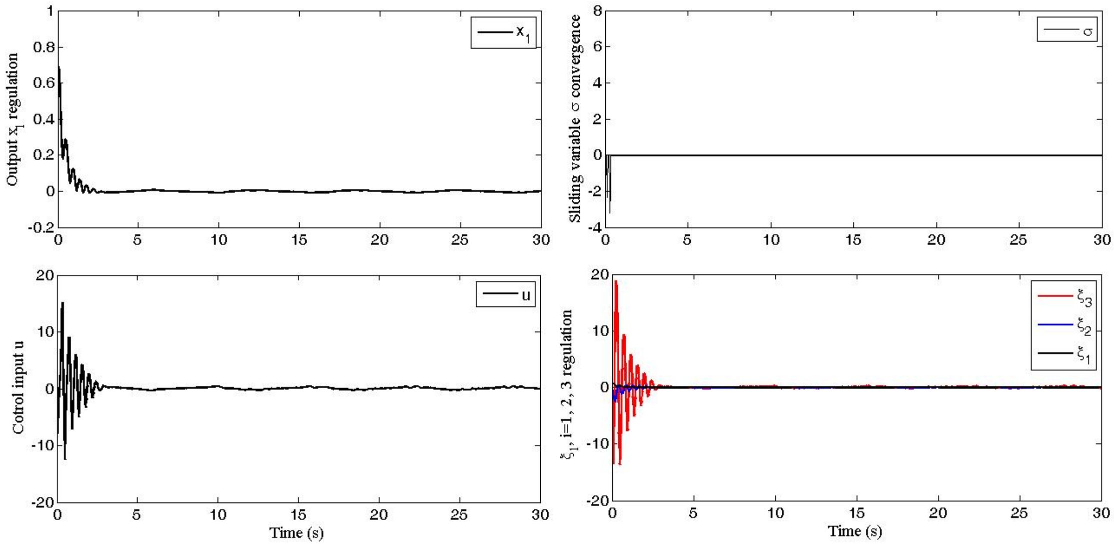


FIGURE 4.3: Output regulation, control effort, sliding variable convergence and $[\xi_1, \xi_2, \xi_3]^T$ regulation in the presence of matched and unmatched uncertainty via the proposed control law.

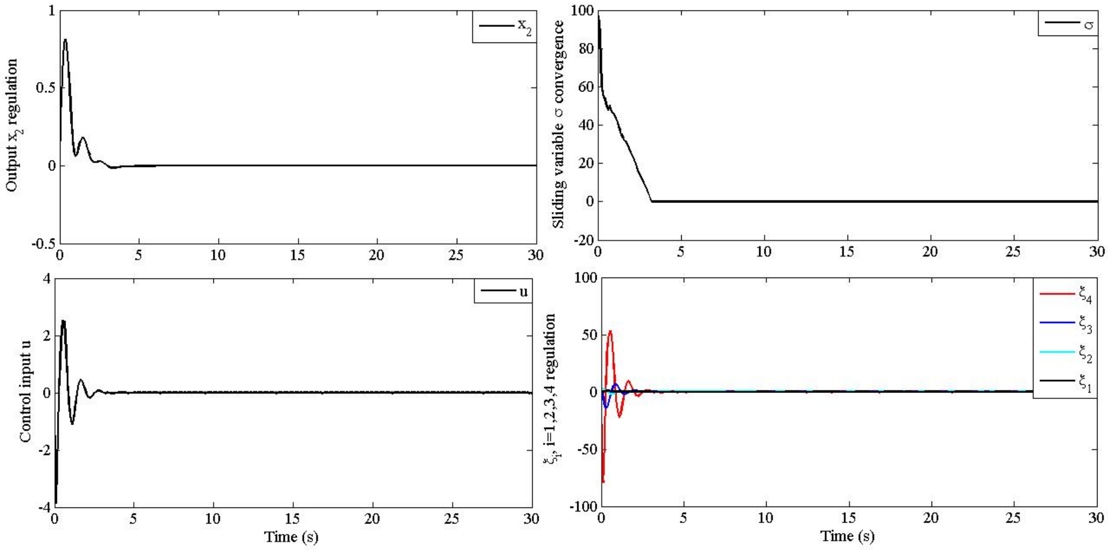


FIGURE 4.4: Output regulation, control effort, sliding variable convergence and $[\xi_1, \xi_2, \xi_3, \xi_4]^T$ regulation in the presence of matched uncertainty via the proposed control law.

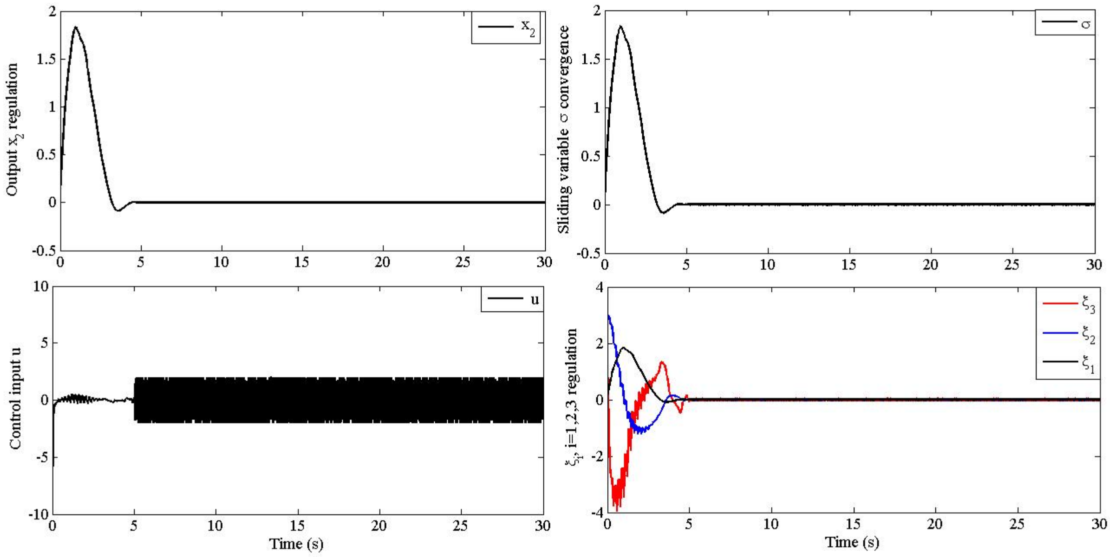


FIGURE 4.5: Output regulation, control effort, sliding variable convergence and $[\xi_1, \xi_2, \xi_3]^T$ regulation in the presence of matched uncertainty via the 3-QCSMC.

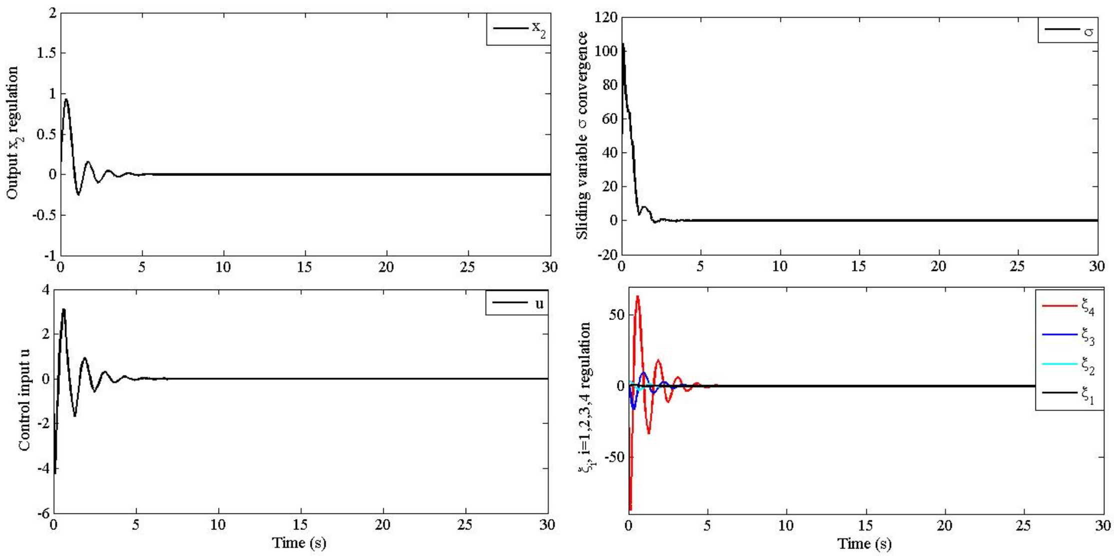


FIGURE 4.6: Output regulation, control effort, sliding variable convergence and $[\xi_1, \xi_2, \xi_3, \xi_4]^T$ regulation in the presence of matched and unmatched uncertainty via the proposed control law.

unmatched state dependent uncertainties. The uncertain system output trajectories are asymptotically regulated to zero in spite of the presence of the uncertainties, while a sliding mode is enforced in finite time along an integral manifold. The use of the integral sliding manifold allows one to subdivide the control design procedure into two steps. First a linear control component is designed by pole placement and then a discontinuous control component is added so as to cope with the uncertainty presence. The design procedure is performed relying on a suitably transformed system. As a consequence, the control acting on the original system is obtained as the output of a chain of integrators and is, accordingly, continuous. This can be a clear benefit in many applications, such as those of mechanical nature, where a discontinuous control action could be non appropriate or even disruptive.

In the next chapter, the methodology presented in chapter 3 and 4 are extended to the case of MIMO systems. In the first case, asymptotic sliding modes are established via the strong reachability conditions while in the second case, enforcement of the sliding mode,in finite time, in the presence of matched and unmatched uncertainties is presented. Numerical examples are given to demonstrate the performance of the systems.

Chapter 5

DISM CONTROL METHODOLOGY FOR NONLINEAR MIMO SYSTEMS

The output feedback control design and their robustness against matched and unmatched uncertainties, for a class of SISO nonlinear systems, is discussed in Chapter 3 and 4. The design procedure and robustness analysis is extended to MIMO nonlinear Systems [21]. The feature which were claimed remains true for these MIMO system. This chapter, in first half, presents the extension of Chapter 3 for these nonlinear systems. The asymptotic stabilization of the output will be ensured when asymptotic sliding modes are established and a couple of numerical examples are presented to explain the control design. However, in the second half, the original problem of the first half, will be considered under states dependent matched and unmatched uncertainties [22]. In addition, the finite time enforcement of sliding mode will be ensured in these uncertainties presence and the asymptotic convergence of the outputs will take place. The uncertainties considered in this study may be states dependent. The output vector of the nonlinear system achieve the desired value under the action of the proposed control law and this claim will be verified via a numerical example. The rest of the chapter is organized as follows. The Section 5.1 contains the problem formulation and Section 5.2 contains the control design procedure. In Section 5.3 a couple of numerical examples are considered and a comprehensive comparative study with DSMC and ISMC is presented. In section 5.4, the problem discussed in Section 5.1, is considered with a class of matched and unmatched uncertainties. Furthermore, the stability of the system with both matched and unmatched uncertainties are theoretically analyzed in Section 5.5. A numerical example is simulated in Section 5.6 to prove the uncertainty compensation/rejection. In the last Section the summary of the chapter is presented.

5.1 Problem Formulation

Consider a MIMO nonlinear system described by a state equation

$$\dot{x} = f(x,t) + g(x,t)u + \zeta(x,t) \qquad \textit{Eq (5.1)}$$

$$y = h(x) \qquad Eq\ (5.2)$$

where $x \in R^n$, $u \in R^m$, f: $R^n \times R^+ \rightarrow R^n$ and g: $R^n \times R^+ \rightarrow R^n$ and h:$R^n \times R^+ \rightarrow R^p$ are smooth vector fields. The term $\zeta(x,t)$ represents the matched uncertainties which may occur due to unmodeled dynamics, parametric variations and external disturbances. Furthermore, it is assumed that the unknown functions $\zeta(x,t)$ are norm bounded i.e., $||\zeta(x,t)|| \leq \zeta_0$, where ζ_0 is some positive constant. The system in *Eq (5.1)* is assumed to be square i.e., $p = m$.

In this Chapter, like Chapter 3, the problem we want to solve (Problem 1) is that of steering the vector of outputs to zero asymptotically i.e., an output regulation problem is considered here in the presence of some uncertainties of matched nature. In order to design the control law, system *Eq (5.1)* is transformed in the following form using some nonlinear transformation (see for instance, [14], [21])

$$\begin{aligned} y_1^{(n_1)} &= \varphi_1(\hat{y}, \hat{u}, u_k^{(\beta k)}) + \gamma_1(\hat{y})u_1^{(\beta 1)} + \zeta_1^*(\hat{y}, t) \\ y_2^{(n_2)} &= \varphi_2(\hat{y}, \hat{u}, u_k^{(\beta k)}) + \gamma_2(\hat{y})u_2^{(\beta 2)} + \zeta_2^*(\hat{y}, t) \\ &\vdots \\ y_p^{(n_p)} &= \varphi_p(\hat{y}, \hat{u}, u_k^{(\beta k)}) + \gamma_p(\hat{y})u_p^{(\beta p),} + \zeta_p^*(\hat{y}, t) \end{aligned} \qquad Eq\ (5.3)$$

where $\hat{y} = [\hat{y}_1, \hat{y}_2, \cdots, \hat{y}_p]^T$, $\hat{y}_i = [y_i, \dot{y}_i, \cdots, y_i^{(n_i-1)}]^T$, $\hat{u} = [\hat{u}_1, \hat{u}_2, \cdots, \hat{u}_p]^T$, $\hat{u}_i = [u_i, \dot{u}_i, \cdots, u_i^{(\beta_i-1)}]^T$, with $\sum_{i=1}^{p} n_i = n$. and βi for $i = 1, 2, \cdots, p$ are some non negative integers which indicate the derivative of the control input. The System in *Eq (5.3)* can be written, in compact form, as follows

$$y_i^{(n_i)} = \varphi_i(\hat{y}, \hat{u}, u_k^{(\beta k)}) + \gamma_i(\hat{y})u_i^{(\beta i)} + \zeta_i^*(\hat{y}, t), i = 1, 2, \cdots, p \qquad Eq\ (5.4)$$

Now by defining the transformation $y_i^{(j-1)} = \xi_{ij}$ for $i = 1, 2, \cdots, p$, $j = 1, 2, ..., n_i$ and $\hat{y}_i = \hat{\xi}_i$, the system *Eq (5.4)* in the LGCC form (as reported in *Eq (3.5)*) and can be written as follows

$$\begin{aligned} \dot{\xi}_{i1} &= \xi_{i2} \\ \dot{\xi}_{i2} &= \xi_{i3} \\ &\vdots \\ \dot{\xi}_{in_i} &= \varphi_i(\hat{\xi}, \hat{u}) + \gamma_i(\hat{\xi})u_i^{(\beta_i)} + \zeta_i^*(\hat{\xi}, t) \end{aligned} \qquad Eq\ (5.5)$$

Furthermore, $\zeta_i^*(\hat{\xi}, t)$ is Lebesgue measurable and satisfy

$$||\zeta_i^*(\hat{\xi}, t)|| \leq \rho||\hat{\xi}|| + l_i \qquad \textit{Eq (5.6)}$$

where $\rho \geq 0$ and $l_i \geq 0$ for $i = 1, 2, \cdots, p$. The system presented in *Eq (5.5)* are p sub-system which have their respective input and output. The output and control input of one subsystem in the other subsystem may treated either as disturbances (coupling effect). The nominal system corresponding to each i^{th} may be obtained by putting each $\zeta_i^*(\hat{\xi}, t) = 0$. Thus, the uncertainty free i^{th} subsystem become

$$\dot{\xi}_{i1} = \xi_{i2}$$

$$\dot{\xi}_{i2} = \xi_{i3}$$

$$\vdots$$

$$\dot{\xi}_{in_i} = \varphi_i(\hat{\xi}, \hat{u}) + \gamma_i(\hat{\xi})u_i^{(\beta_i)} = \Phi_i(\hat{\xi}, \hat{u}, u_i^{\beta_i}, u_k^{(\beta_k)}, t) \qquad \textit{Eq (5.7)}$$

The Definition 3.1 and 3.2 for the aforementioned MIMO nonlinear system become[16]

Definition 5.1. The system in *Eq (5.7)* is called a proper system if

- $p = m$
- $\Phi_i(\hat{\xi}, \hat{u}, u_i^{(\beta_i)}, u_k^{(\beta_k)}, t) \in C^1$
- $det \left[\frac{\partial(\Phi_1, \Phi_2, \cdots, \Phi_p)}{\partial(u_1^{(\beta_1)}, u_2^{(\beta_2)}, \cdots, u_p^{(\beta_p)})} \right] \neq 0$

The definition 3.2 of zero dynamics can be extended to the following form

Definition 5.2. The zero dynamics of the system *Eq (5.7)* are defined by

$$\Phi_i(0, \hat{u}, u_i^{(\beta_i)}, u_k^{(\beta_k)}, t) = 0, i = 1, 2, \cdots, p \qquad \textit{Eq (5.8)}$$

The system in *Eq (5.7)* is called minimum phase if the dynamic in *Eq (5.8)* are uniformly asymptotically stable and weakly minimum phase if these dynamics are marginally stable. The subsequent control strategy is a best candidate for minimum phase as well as for weakly minimum phase systems.

Assumption 5. *The system Eq (5.7) is proper and minimum phase according to Definitions 5.1 and 5.2, respectively.*

Note that, the components of the control input u for system in *Eq (5.1)* can be obtained as the solution of the following differential equation

$$\begin{aligned} \dot{u}_{i1} &= u_{i2} \\ \dot{u}_{i2} &= u_{i3} \\ &\vdots \\ \dot{u}_{in_i} &= \Psi_i'(\hat{\xi}, \hat{u}, u_k^{(\beta_k)}, t), 1 \leq k \leq m \end{aligned} \qquad \text{Eq (5.9)}$$

The function $\Psi_i(\hat{\xi}, \hat{u}, u_k^{(\beta_k)}, t)$ is discontinuous in nature and represents the final expression of the dynamic controller for system referred in *Eq (5.5)*.

Now the original control problem (Problem 1) can be reformulated with reference to system *Eq (5.5)* and to the uncertainty bounds defined in *Eq (5.6)* under the Assumption 5, and to the nominal system in *Eq (5.7)*. Therefore, the new control problem (Problem 2) is to regulate to zero asymptotically the vector of outputs $\hat{\xi} = [\hat{\xi}_1, \hat{\xi}_2, ..., \hat{\xi}_n]^T$ in the presence of matched and unmatched uncertainties. In other words, a regulation problem is considered here. The solution to (Problem 2) is a clear solution to (Problem 1) since $y = [\xi_{11}, \xi_{21}, ..., \xi_{n1}]^T$.

Now, the design of the proposed control law can be elaborated in the following study.

5.2 Control Design

Like the design procedure which has been presented in chapter 3, this dynamic controller methodology contains a dynamic continuous control component and a dynamic discontinuous control component. The proposed control law for the system *Eq (5.7)* is of dynamic nature which can be expressed as

$$u_i^{(\beta_i)} = u_{0i}^{(\beta_i)} + u_{01}^{(\beta_i)} \qquad Eq\ (5.10)$$

The first part $u_{0i}^{(\beta_i)}$ is continuous which stabilizes the system at the equilibrium point when sliding mode is enforced and the second part $u_{01}^{(\beta_i)}$ is discontinuous in nature which is named as the dynamic integral control. This effectively rejects the uncertainties. In the next two subsections, the design of these components will be discussed.

5.2.1 Design of the Continuous Component $u_{0i}^{(\beta_i)}$

To facilitate the design of this part, the system *Eq (5.7)* can be expressed in an alternative form as follows

$$\begin{aligned}\dot{\xi}_{i1} &= \xi_{i2}\\ \dot{\xi}_{i2} &= \xi_{i3}\\ &\vdots\\ \dot{\xi}_{in_i} &= \chi_i(\hat{\xi}, \hat{u}, u_i^{(\beta_i)}, u_k^{(\beta_k)}) + u_i^{(\beta_i)}\end{aligned} \qquad Eq\ (5.11)$$

where $\chi_i(\hat{\xi}, \hat{u}, u_i^{(\beta_i)}, u_k^{(\beta_k)}) = \varphi_i(\hat{\xi}, \hat{u}, u_k^{(\beta_k)}) + (\gamma_i(\hat{\xi}) - 1)u_i^{(\beta_i)}$. In the design of $u_{0i}^{(\beta_i)} \in R$, system in *Eq (5.11)* is first considered to be independent of nonlinearities and uncertainties i.e.,$\chi_i(\hat{\xi}, \hat{u}, u_i^{(\beta_i)}) = 0$ and $\zeta_i^*(\hat{\xi}, t) = 0$ and it is also supposed to operate under the continuous control component $u_{0i}^{(\beta_i)}$ only. Then, the system *Eq (5.11)* becomes

$$\begin{aligned}\dot{\xi}_{i1} &= \xi_{i2}\\ \dot{\xi}_{i2} &= \xi_{i3}\\ &\vdots\\ \dot{\xi}_{in_i} &= u_{0i}^{(\beta_i)}\end{aligned} \qquad Eq\ (5.12)$$

This is an i^{th} linear subsystem which can also be expressed in the forth coming general form

$$\dot{\hat{\xi}}_i = A_i\hat{\xi}_i + B_i u_{0i}^{(\beta_i)} \qquad Eq\ (5.13)$$

where $A_i = \begin{bmatrix} O_{(n_i-1)\times 1} & I_{(n_i-1)\times(n_i-1)} \\ O_{1\times 1} & O_{1\times(n_i-1)} \end{bmatrix}$ and $B_i = \begin{bmatrix} O_{(n_i-1)\times 1} \\ I_{1\times 1} \end{bmatrix}$. and each $\hat{\xi}_i \in R^{n_i}$ is the state vector of the i^{th} subsystem. This control component can be designed via pole placement and it will appear in the following form

$$u_{0i}^{(\beta_i)} = K_i^T \hat{\xi}_i \qquad \textit{Eq (5.14)}$$

The total input output form for the p subsystems becomes

$$\dot{\hat{\xi}} = \widetilde{A}\hat{\xi} + \widetilde{B}\widetilde{u} \qquad \textit{Eq (5.15)}$$

where $\widetilde{A} = diag\,[A_1, A_2, \cdots, A_p]$, $\widetilde{B} = diag\,[B_1, B_2, \cdots, B_p]$, and $\widetilde{u} = [u_1, u_2, \cdots, u_p]^T$

5.2.2 Design of the Discontinuous Component $u_{01}^{(\beta_i)}$

The discontinuous control design methodology is exactly the same which is presented in Section 3.2.2. The construction is being pursued by first defining the integral manifold which has the following mathematical representation

$$\sigma_i(\hat{\xi}_i) = \sigma_{0i}(\hat{\xi}_i) + z_i, i = 1, 2, \cdots, p \qquad \textit{Eq (5.16)}$$

The first term on the right hand side of the *Eq (5.16)* is the Hurwitz polynomial (conventional sliding manifold) and the second is integral term which is used in order to help in the elimination of the reaching phase and to provide more stability. The conventional sliding surface for MIMO system becomes $\sigma_{0i}(\hat{\xi}) = \sum_{i=1}^{n_i} c_{il}\xi_{il}$ with $c_{in_i} = 1$. The time derivative of *Eq (5.16)* along with the dynamics of the system in *Eq (5.7)* or *Eq (5.11)*, yields

$$\dot{\sigma}_i = \sum_{i=1}^{(n_i-1)} c_{il}\xi_{il+1} + \chi_i(\hat{\xi}, \hat{u}, u_i^{(\beta_i)}) + u_{0i}^{(\beta_i)} + u_{1i}^{(\beta_i)} + \dot{z} \qquad \textit{Eq (5.17)}$$

Choosing the integral term with the following mathematical expression

$$\dot{z} = -\left[\sum_{i=1}^{(n_i-1)} c_{il}\xi_{il+1} + u_{0i}^{(\beta_i)}\right] \qquad \textit{Eq (5.18)}$$

The initial conditions of $z_i(0)$ are selected in such a way that satisfy the requirement $\sigma_i(0) = 0$.

$$\dot{\sigma}_i = \varphi_i(\hat{\xi}, \hat{u}) + (\gamma_i(\hat{\xi}) - 1)u_{0i}^{(\beta_i)} + \gamma_i(\hat{\xi})u_{1i}^{(\beta_i)} \qquad Eq\ (5.19)$$

Now, consider the extended decoupled form of the reachability condition expressed in *Eq (3.15)* along with the satisfaction of Definition 2.3.

$$\dot{\sigma}_i(\xi) = -K_i\sigma_i - K_{0i}sign(\sigma_i) \qquad Eq\ (5.20)$$

The comparison of *Eq (5.19)* and *Eq (5.20)* results in the following dynamic discontinuous control component

$$u_{1i}^{\beta_i} = -\frac{1}{\gamma_i(\hat{\xi})}\left[\varphi_i(\hat{\xi}, \hat{u}) + (\gamma_i(\hat{\xi}) - 1)u_{0i}^{(\beta_i)} + K_i\sigma_i + K_{0i}sign(\sigma_i)\right] \qquad Eq\ (5.21)$$

The constants K_i and K_{0i} are control gains which are selected according to uncertainty bounds [5](chapte 5), which satisfy the *Eq (5.6)*. This control law enforces sliding mode asymptotically along the manifold *Eq (5.16)* from the very beginning. This completes the design of the discontinuous control component. The final controller can be obtained by just inserting *Eq (5.14)* and *Eq (5.21)* in *Eq (5.10)*. Thus, the final expression of the proposed control law takes the form

$$u_i^{\beta_i} = u_{0i}^{\beta_i} - \frac{1}{\gamma_i(\hat{\xi})}\left[\varphi_i(\hat{\xi}, \hat{u}) + (\gamma_i(\hat{\xi}) - 1)u_{0i}^{(\beta_i)} + K_i\sigma_i + K_{0i}sign(\sigma_i)\right] \qquad Eq\ (5.22)$$

Note that this control law can be implemented by integrating the derivative of the control, $u_{1i}^{(\beta_i)}$, (β_i) times (i.e., after the solution of the differential equation reported in *Eq (5.9)*.) so that the control input actually applied to the system is continuous. This can be a benefit for various class of systems such as those of mechanical type, for which a discontinuous control action could be disruptive. The reachability law used in this control law results in asymptotic sliding modes, therefore, the following stability analysis contains a theorem which will discuss about the asymptotic sliding mode enforcement.

5.2.3 Stability Analysis

The convergence of the plant output to the desired output can be guaranteed when sliding mode is ensured. Therefore, the following theorem is stated:

Theorem 5.3. *Consider the nonlinear system in Eq (5.5), subject to Assumption 5 and the uncertainties bounds in Eq (5.6), if the sliding surface, discontinuous control law $u_{1i}^{\beta_i}$ and the integral term dynamics are chosen according Eq (5.16), Eq (5.21) and Eq (5.18) respectively, then the convergence condition is satisfied.*

Proof: Let consider a Lyapunove's candidate for each i^{th} subsystem

$$v_i = \frac{1}{2}\sigma_i^2, i = 1, 2, \cdots, p \qquad \text{Eq (5.23)}$$

The time derivative of *Eq (5.23)* along *Eq (5.11)*, yields

$$\dot{v}_i = \sigma_i \left[\sum_{i=1}^{(n_i-1)} c_{il}\xi_{il+1} + \chi_i(\hat{\xi}, \hat{u}, u_i^{(\beta_i)}) + u_{0i}^{(\beta_i)} + u_{1i}^{(\beta_i)} + \zeta_i^*(\hat{\xi}, t) + \dot{z} \right] \qquad \text{Eq (5.24)}$$

Substituting the value of the integral term from *Eq (5.18)* into *Eq (5.24)*, yields

$$\dot{v}_i = \sigma_i \left[\chi_i(\hat{\xi}, \hat{u}, u_i^{(\beta_i)}) + u_{1i}^{(\beta_i)} + \zeta_i^*(\hat{\xi}, t) \right] \qquad \text{Eq (5.25)}$$

Using *Eq (5.21)* in *Eq (5.25)*, while keeping in view *Eq (5.6)* and the uncertainties bounds criterion reported in [16], one has

$$\dot{v}_i \leq -\sigma_i \left[K_i\sigma_i + K_{0i} sign(\sigma_i) \right] \qquad \text{Eq (5.26)}$$

This confirms the stability of the i^{th} subsystem. The stability of the nonlinear system working under the p-inputs and p-outputs can be ensured by defining the Lyapunov function

$$V = \frac{1}{2}\sigma^T\sigma \qquad \text{Eq (5.27)}$$

The derivative of *Eq (5.27)* with use of *Eq (5.26)* reduces to the following form

$$\dot{V} \leq \sum_{i=1}^{p} v_i \qquad \text{Eq (5.28)}$$

Therefore, it is confirmed that the sliding manifold $\sigma_i = 0$ is reached for every subsystem and consequently, sliding mode exists for the overall dynamic system *Eq (5.5)*. This completes the proof.

5.3 Illustrative examples

Design algorithm presented in Section 5.2 has been applied to design controller for Three Tank System and a Nonlinear System. A comparative analysis of DISMC, DSMC and Integral Sliding Mode Controller (ISMC), is put forwarded in the following two subsections. The assessment of controllers is carried out on the basis of output convergence, sliding manifold convergence and controller with alleviated chattering effort in the presence of uncertainties.

5.3.1 Example 1: Nonlinear system

This example is adapted from the theory of DSMC theory publication [16]. The standard results of DSMC presented in the aforementioned work will be compared with this proposed method (DISMC) to authenticate the proposed methodology advantages.

5.3.1.1 System Description in LGCCF

Consider the MIMO nonlinear system with the following LGCCF description [16]

$$\dot{\xi}_1 = \xi_2$$

$$\dot{\xi}_2 = -\xi_1\xi_2 - \xi_3 u_1 - \xi_2 u_2^2 - \dot{u}_1 - u_1^3 + \lambda(t)u_2 + u_1 sin(u_1) + cos(x_3)$$

$$\dot{\xi}_3 = \xi_4$$

$$\dot{\xi}_4 = -\xi_2\xi_3^2 - \xi_1 u_2^2 + \xi_4^2 u_1 - \dot{u}_2 - u_2^3 - \lambda(t)u_1 - u_2 sin(u_2)$$

where $\hat{\xi_1} = [\xi_1, \xi_2]^T$ and $\hat{\xi_2} = [\xi_3, \xi_4]^T$ are the state vectors of the two subsystems. The objective is to regulate the states of each subsystem to the equilibrium point.

The zero dynamics of this system have the forthcoming mathematical expressions.

$$\dot{u}_1 = -u_1^3 + \lambda(t)u_2 + u_1 sin(u_1)$$

$$\dot{u}_2 = -u_2^3 - \lambda(t)u_1 - u_2 sin(u_2)$$

These zero dynamics are Lagrange stable for different Lebesgue measurable $\lambda(t)$. In the forthcoming study of this example, λ is set equal to 1.

5.3.1.2 Control Law Design

The control law design is initiated by defining the integral sliding surface in the following form

$$\sigma_1 = a_{11}\xi_1 + \xi_2 + z_1$$

$$\sigma_2 = a_{21}\xi_3 + \xi_4 + z_2$$

where z_1 and z_2 are the integral terms. In the above LGCCF form it is clear that

$$\varphi_1(\hat{\xi}, t) = a_{11}\xi_2 + \left(-\xi_1\xi_2 - \xi_3 u_1 - \xi_2 u_2^2 - u_1^3 + \lambda(t)u_2 + u_1 sin(u_1)\right)$$

$$\varphi_2(\hat{\xi}, t) = a_{21}\xi_4 + \left(-\xi_2\xi_3^2 - \xi_1 u_2^2 + \xi_4^2 u_1 - u_2^3 - \lambda(t)u_1 - u_2 sin(u_2)\right)$$

$$\gamma_1(\hat{\xi}) = \gamma_2(\hat{\xi}) = -1$$

The expressions of the linear dynamic controllers and integral term dynamics takes the form

$$\dot{u}_{10} = k_1\xi_1 + k_2\xi_2$$

$$\dot{u}_{20} = k_3\xi_3 + k_4\xi_4$$

$$\dot{z}_1 = \dot{u}_{10}$$

$$\dot{z}_2 = \dot{u}_{20}$$

The discontinuous control laws is designed while taking the strong coupled reachablity condition. By coupling of the reachability we mean that the reachability depends on both the integral sliding surfaces. The control law expression takes

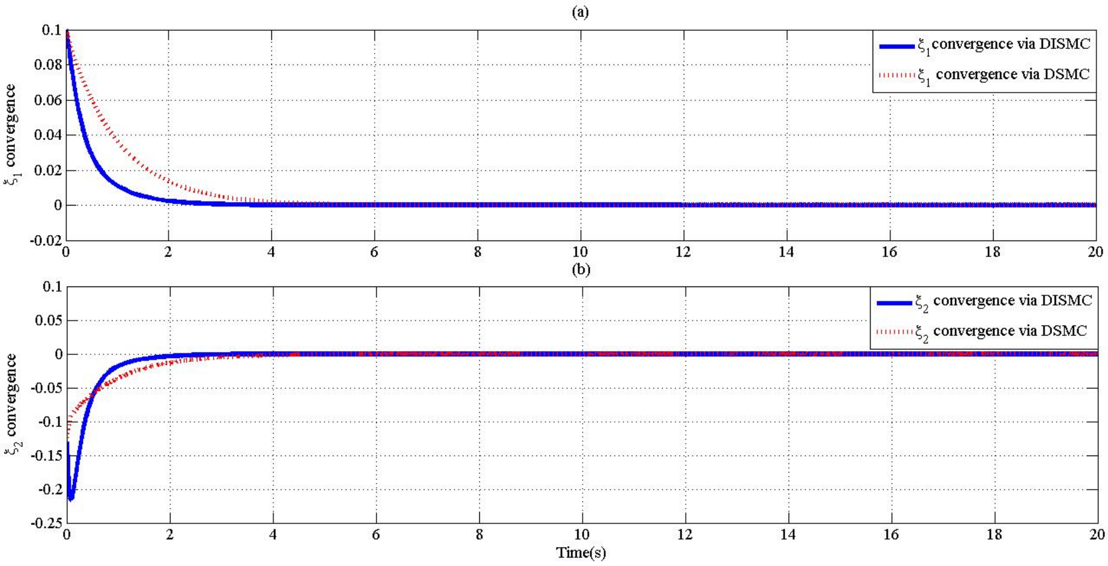
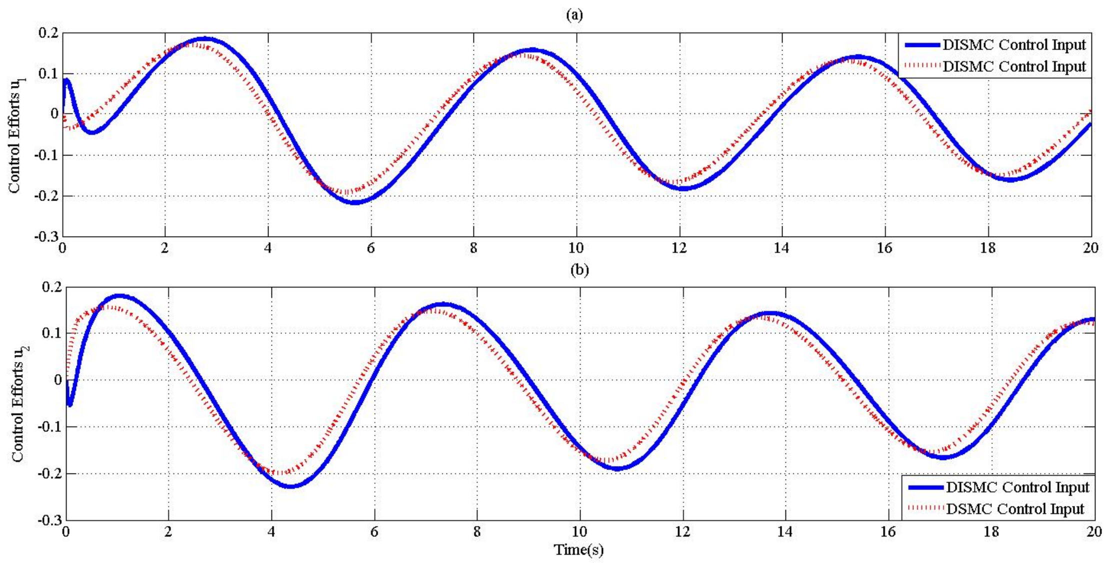


FIGURE 5.1: (a). ξ_1 convergence via DISMC (solid line) and DSMC (doted line). (b). ξ_2 convergence via DISMC (solid line) and DSMC (doted line).

the following form

$$\dot{u}_{11} = \varphi_1(\hat{\xi}, t) - \dot{u}_{10} + K_{11}\sigma_1 + K_{12}\sigma_2 + K_{01}sign(\sigma_1)$$

$$\dot{u}_{21} = \varphi_2(\hat{\xi}, t) - \dot{u}_{20} + K_{21}\sigma_1 + K_{22}\sigma_2 + K_{02}sign(\sigma_2)$$



(a). Control Input signal u_1 via DISMC (solid line) and DSMC (doted line). (B).

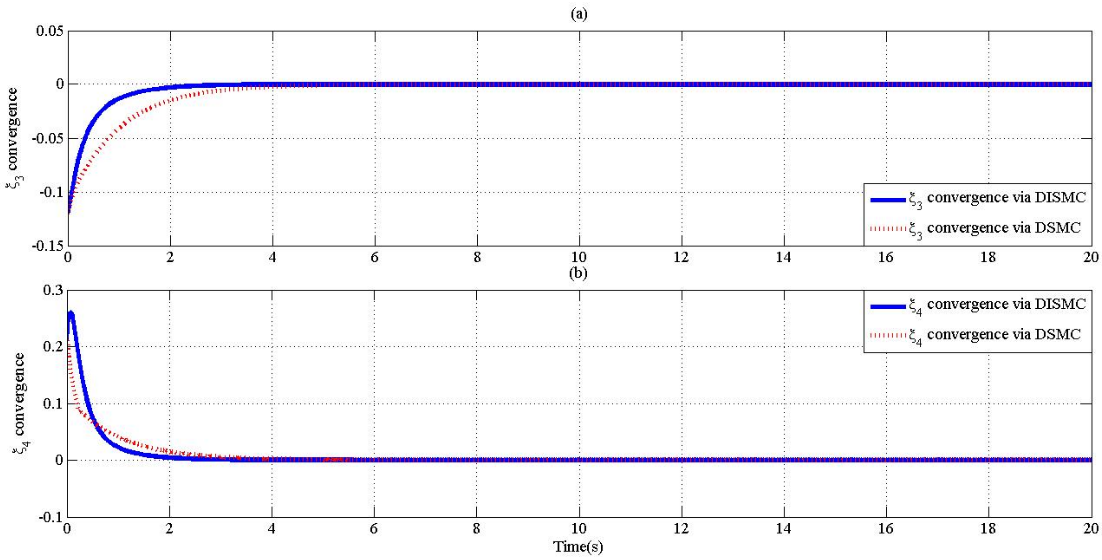


FIGURE 5.2: (a). ξ_3 convergence via DISMC (solid line) and DSMC (doted line). (b). ξ_4 convergence via DISMC (solid line) and DSMC (doted line).

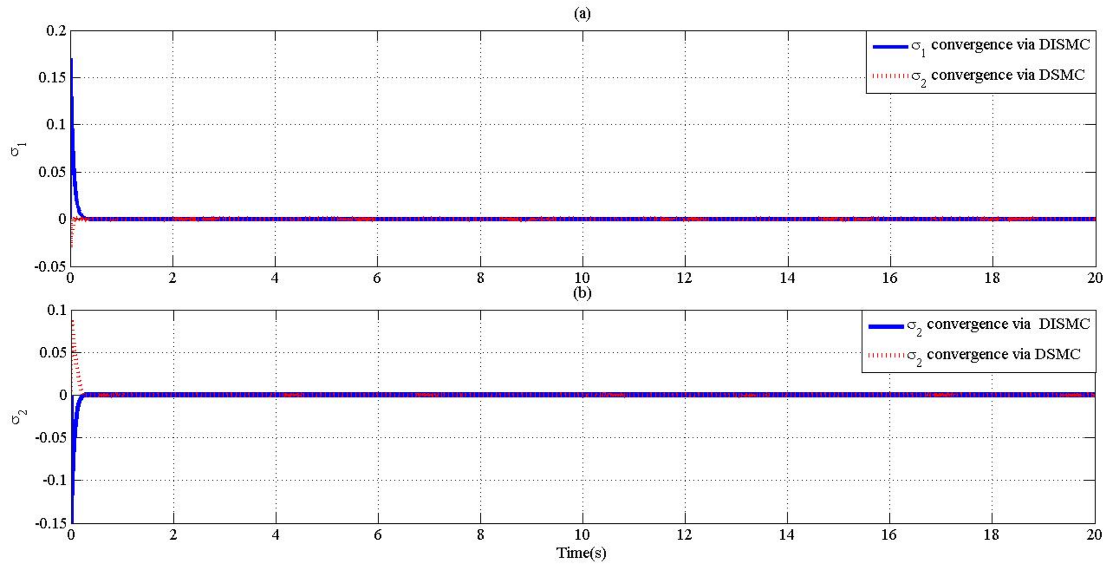


FIGURE 5.3: (a). Sliding surface σ_1 convergence via DISMC (solid line) and DSMC (doted line).

Control Input u_2 signal via DISMC (solid line) and DSMC (doted line).

TABLE 5.1: Gains of the control law u_1 used for both DISMC and X.Y.Lu DSMC controllers.

Parameters	k_1	k_2	K_{11}	K_{12}	K_{01}	a_{11}
DISMC u_1	6.53	2.95	18	1	0.05	3
DSMC u_1	.. .	...	8	1	.05	1

TABLE 5.2: Gains of the control law u_2 used for both DISMC and X.Y.Lu DSMC controllers.

Parameters	k_3	k_4	K_{21}	K_{22}	K_{02}	a_{21}
DISMC u_1	6.53	2.95	2	18	0.04	3
DSMC u_1	...	...	2	8	0.4	1

5.3.1.3 Comparison of Simulation Results

The control objective, in this example, is to regulate the system trajectories to the equilibrium points. Furthermore, the evaluation of the control algorithm in this example is carried out on the basis of the finite settling time of the system states. The convergence of the system state via the new developed control scheme DISMC and standard DSMC scheme is shown in the Figures 5.1 and 5.2. The results of DSMC is produced with the same gains as mentioned in the example 2 of manuscript [16]. The figures show that the response of the new developed control scheme is better than the standard DSMC technique. The control input u_1 and u_2 is displayed in Figure 5.3.1.2. The control inputs of both the technique are chattering free and are similar in pattern just with a little difference is phase and magnitude. Similarly, the sliding surface convergence also ends with the same conclusion as depicted in Figure 5.3. The gains used in this case study are listed in the Table 5.1 and Table 5.2. The gains of DSMC are standard which are used by [16]. Based on the results shown above, It is decided that the new output feedback control law has superiority over DSMC. The next example is also a best candidate for this control scheme.

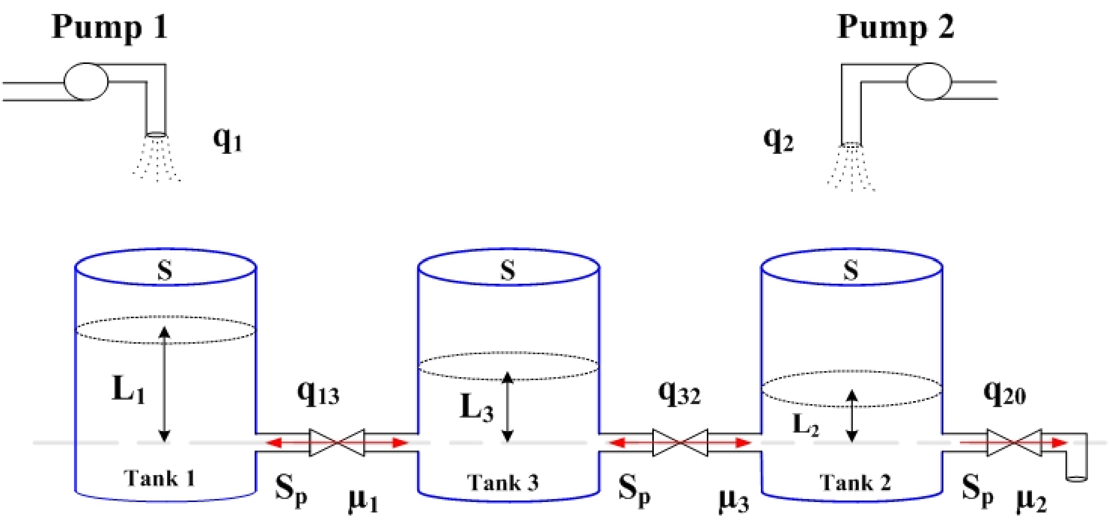


FIGURE 5.4: Schematic Diagram of the Three Tank System

5.3.2 Example 2:MIMO Three Tank System

TABLE 5.3: Typical Parameter Values of Benchmark TTS.

Parameters	Description	Nominal Values	Units
S	Surface area of tanks	0.0154	m^2
S_p	Surface area of the pipe	0.00005	m^2
u_{1max}, u_{2max}	Input Flow Rates	100	ml/s
$y_i, i = 1,2,3$	Maximum level in the tanks	0.62	m
μ_1	Viscosity or flow rates	0.5	...
μ_2	,,	0.675	...
μ_3	,,	0.5	...

5.3.2.1 System Description

The Three Tank System is extensively used for nonlinear controller analysis and represents a typical system in process industry, fuel management system in aircrafts and flight vehicles. A Three Tank System, as depicted in Figure 5.4, contains three interconnected tanks with same surface area S . The terms q_{ij} represents the water flow rates from tank i to j., which, is given by

$$q_{ij} = \mu_i S_p sign(L_i - L_j)\sqrt{2g|L_i - L_j|}, i, j = 1, 2, 3 \qquad Eq\ (5.29)$$

and

$$q_{20} = \mu_2 S_p \sqrt{2gL_2} \qquad \textit{Eq (5.30)}$$

The system parameters μ_i, S_p and S are the flow coefficients, cross sectional areas of the interconnecting pipes and surface area of the tanks. L_i, q_1 and q_2 are the liquids levels in the tanks, flow rates into tank 1 and tank 2, respectively. The full model of the system in state space [69], is given by

$$\begin{aligned}
\dot{x}_1 &= -C_1 sign(x_1 - x_3)\sqrt{x_1 - x_3} + \frac{u_1}{S} \\
\dot{x}_2 &= C_3 sign(x_3 - x_2)\sqrt{x_3 - x_2} - C_2 sign x_2 \sqrt{x_2} + \frac{u_2}{S} \qquad \textit{Eq (5.31)} \\
\dot{x}_3 &= C_1 sign(x_1 - x_3)\sqrt{x_1 - x_3} - C_3 sign(x_3 - x_2)\sqrt{x_3 - x_2}
\end{aligned}$$

where $x_i(t)$ is the liquid level in the Tank i and $C_i = \frac{1}{s}\mu_2 S_p \sqrt{2g}$ are derived parameters. The control signals u_1 and u_2 are input flow rates, respectively. The typical parameters values of three tank system are given in Table 5.3. For the sake of clarity, it is supposed that $x_1 > x_3 > x_2$, then the equations of motion*Eq (5.31)*, can be written as follows:

$$\begin{aligned}
\dot{x}_1 &= -C_1 \sqrt{x_1 - x_3} + \frac{u_1}{S} \\
\dot{x}_2 &= C_3 \sqrt{x_3 - x_2} - C_2 \sqrt{x_2} + \frac{u_2}{S} \qquad \textit{Eq (5.32)} \\
\dot{x}_3 &= C_1 \sqrt{x_1 - x_3} - C_3 \sqrt{x_3 - x_2}
\end{aligned}$$

The control objective in three tank system is to maintain a certain tank level under disturbances and parametric variations.

5.3.2.2 Controller Design

In order to achieve the above task, the proposed methodology is employed. The outputs of interest are $y_1 = x_1$ and $y_2 = x_2$, then *Eq (5.32)*, in the I-O form can be rewritten as

$$\begin{aligned}
\dot{\xi}_{11} &= \xi_{12} \\
\dot{\xi}_{12} &= \phi_1(\hat{\xi}_1, \hat{u}_1, \dot{u}_1)
\end{aligned}$$

$$\dot{\xi}_{21} = \phi_2(\hat{\xi}_2, u_2)$$

where $\phi_1(\hat{\xi}_1, \hat{u}_1, \dot{u}_1) = \frac{-C_1}{2\sqrt{|x_1-x_3|}}\left[-2\sqrt{|x_1-x_3|} + C_3\sqrt{x_3-x_2} + \frac{u_1}{S}\right] + \frac{\dot{u}_1}{S}$
$\phi_2(\hat{\xi}_2, u_2) = C_3\sqrt{|x_3-x_2|} - C_2\sqrt{|x_2|} + \frac{u_2}{S}$,
$\hat{\xi}_1 = [\xi_{11}, \dot{\xi}_{11}]^T$, $\hat{u}_1 = u_1$ and $\hat{\xi}_2 = \xi_{21}$. In addition, Definitions 5.1 and 5.2 are satisfied. The transformations for the above system are of the following form

$$\xi_{11} = x_1$$

$$\xi_{12} = -C_1\sqrt{x_1-x_3} + \frac{u_1}{S}$$

$$\xi_{21} = x_2$$

Thus, $\hat{\xi}_1 = [\xi_{11}, \xi_{12}]^T$, and $\hat{\xi}_2 = \xi_{21}$ are the state vectors of the two subsystems. The corresponding linear subsystems becomes

$$\dot{\hat{\xi}}_1 = A_1\hat{\xi}_1 + B_1\dot{u}_{01}$$

$$\dot{\hat{\xi}}_2 = A_2\hat{\xi}_2 + B_2 u_{02}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \textit{Eq (5.33)}$$

and

$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \textit{Eq (5.34)}$$

where $A_2 = [0]$ and $B_2 = [1]$. The solution of Riccatti equation being mentioned in chapter 3 section 3.2.1 appears in the form of the following matrices

$$P_1 = \begin{bmatrix} 1.0311 & 0.0316 \\ 0.0316 & 0.0326 \end{bmatrix}$$

$$P_2 = [0.0316]$$

and the linear controllers becomes

$$\dot{u}_{01} = -k_{11}\xi_{11} - k_{12}\xi_{12}$$

$$u_{02} = -k_{21}\xi_{21}$$

Now, the nonlinear discontinuous controllers are designed by defining the integral sliding surfaces as follows

$$s_1 = c_{11}\xi_{11} + \xi_{12} + z_1$$

$$s_2 = c_{21}\xi_{21} + z_2$$

Following the methodology of of the discontinuous control law design presented in the previous Section, the expressions of integral terms are given by

$$\dot{z}_1 = -c_{11}\left[-C_1\sqrt{|x_1 - x_3|} + \frac{u_1}{S} + \frac{\dot{u}_{01}}{S}\right]$$

$$\dot{z}_2 = -c_{21}\left[C_3\sqrt{|x_3 - x_2|} - C_2\sqrt{|x_2|} + \frac{u_{02}}{S}\right]$$

and the discontinuous parts are obtained with the following expressions

$$\dot{u}_{11} = \frac{C_1}{2\sqrt{|x_1 - x_3|}}\left[-2\sqrt{|x_1 - x_3|} + C_3\sqrt{x_3 - x_2} + \frac{u_1}{S}\right] - S\left[K_1 s_1 + K_{01} sign(s_1)\right]$$

$$u_{21} = -S\left[K_2 s_2 + K_{02} sign(s_2)\right]$$

The final expression of the controllers appear as follows

$$\begin{aligned}\dot{u}_1 = & -k_{11}\xi_{11} - k_{12}\xi_{12} - \frac{C_1}{2\sqrt{|x_1 - x_3|}}\left[-2\sqrt{|x_1 - x_3|} + C_3\sqrt{x_3 - x_2} + \frac{u_1}{S}\right] \\ & - S\left[K_1 s_1 + K_{01} sign(s_1)\right]\end{aligned}$$

$$u_2 = -k_{21}\xi_{21} - S\left[K_2 s_2 + K_{02} sign(s_2)\right]$$

The above two expressions will stabilize the respective outputs and can be implemented by passing through a low pass filter(integrator) which will eliminate the high amplitude unwanted peaks being caused by switching imperfections.

TABLE 5.4: Gains of the control law $\dot{u}_1, u_2$ used for DISMC simulations in the presence of output additive uncertainties.

Parameters	k_{11}	k_{12}	k_{21}	K_1, K_2	K_{01}, K_{02}	c_{11}	c_{12}	c_{21}
DISMC u_1	31.6	32.6	1	10	0.001	25	1	6.5

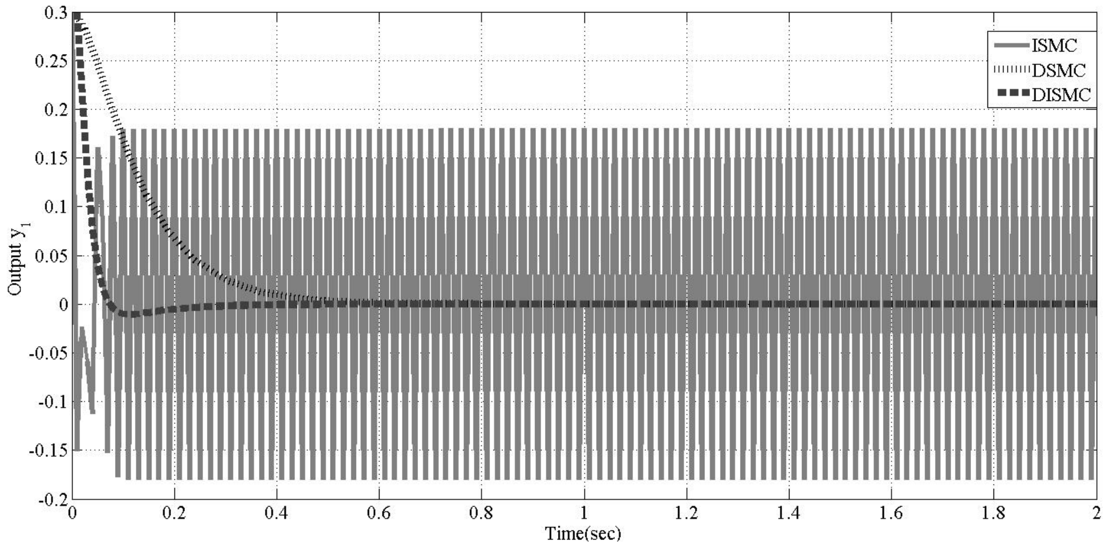


FIGURE 5.5: y_1 trajectories in the presence of uncertainty via ISMC, DSMC and DISMC

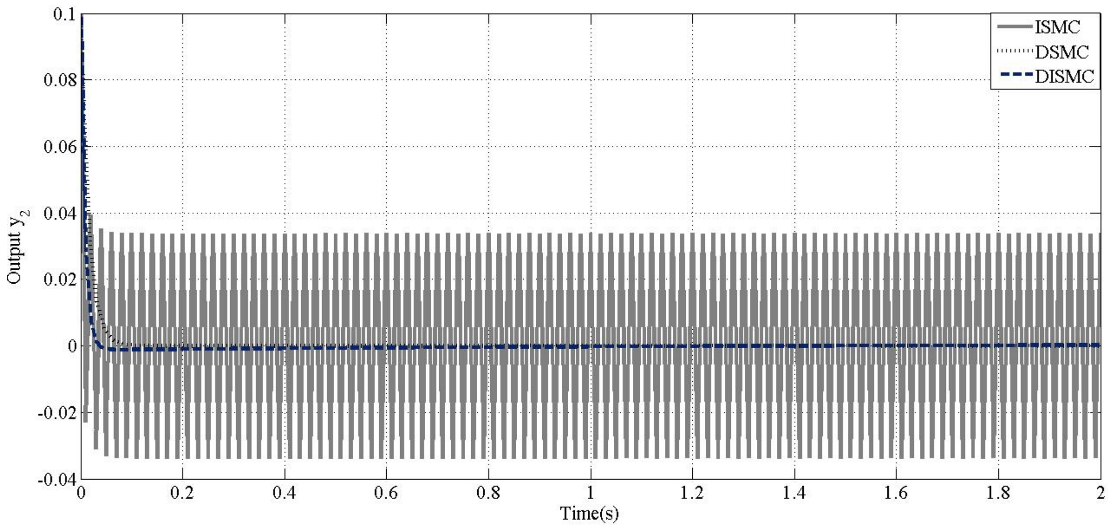


FIGURE 5.6: y_2 trajectories in the presence of uncertainty via ISMC, DSMC and DISMC.

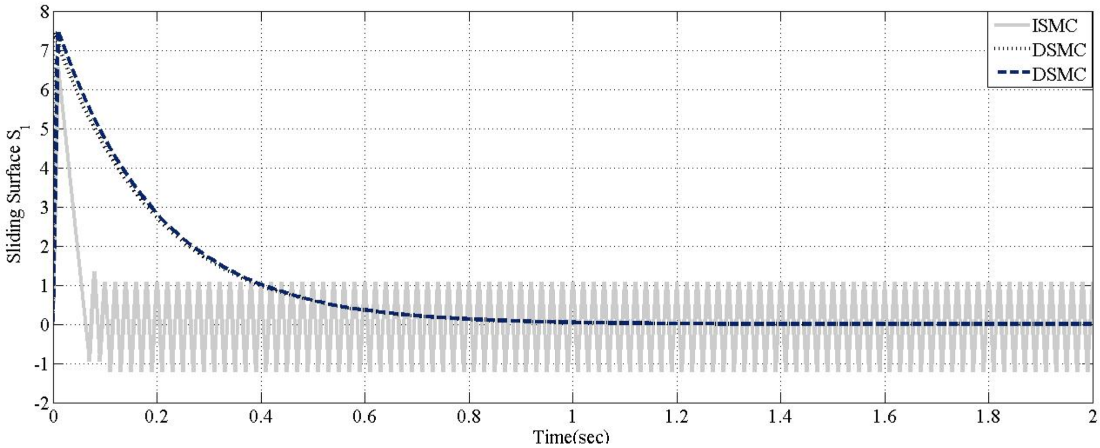


FIGURE 5.7: Sliding Surface σ_1 convergence for ISMC, DSMC and DISMC in the presence of uncertainty.

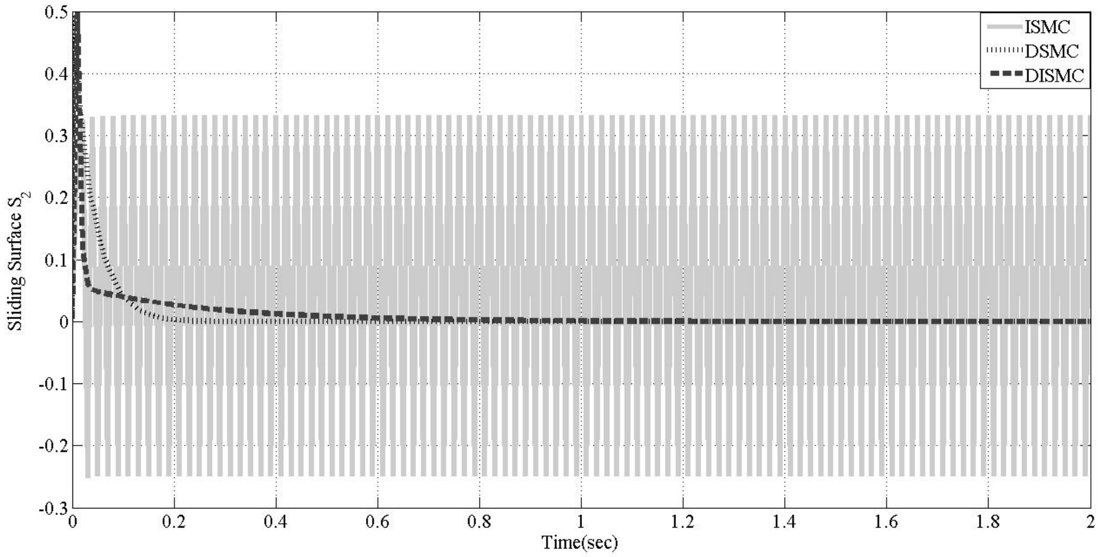


FIGURE 5.8: Sliding Surface σ_2 convergence for ISMC, DSMC and DISMC in the presence of uncertainty

5.3.3 Simulation Results Comparison

The evaluation of the proposed control algorithm is carried out in the presence of parametric variations and state dependent uncertainties in three tank system. The comparative analysis of ISMC, DSMC and DISMC is executed in the presence of these uncertainties.

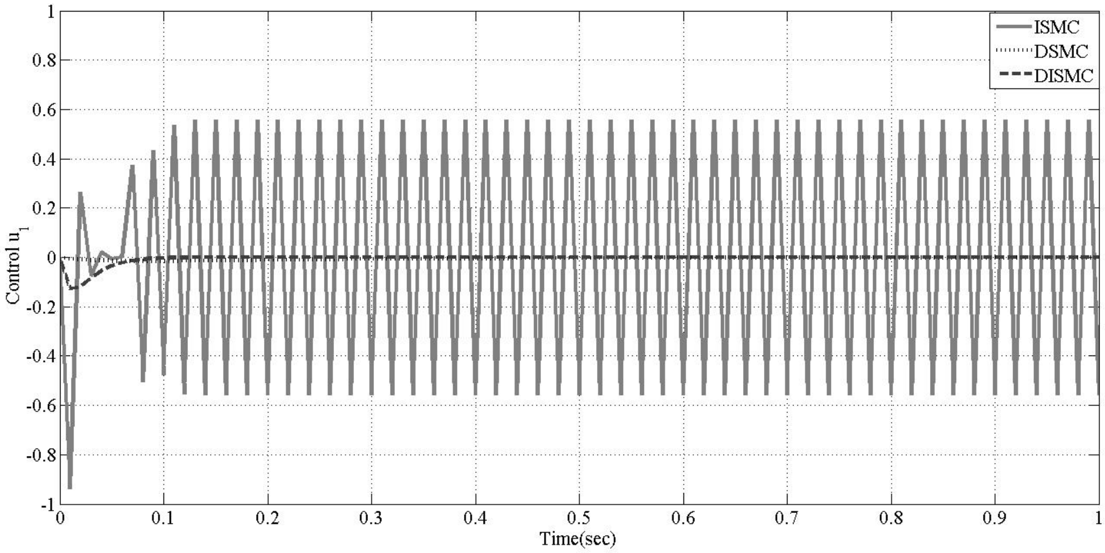


Figure 5.9: Control Effort u_1 via ISMC, DSMC and DISMC in the presence of uncertainty.

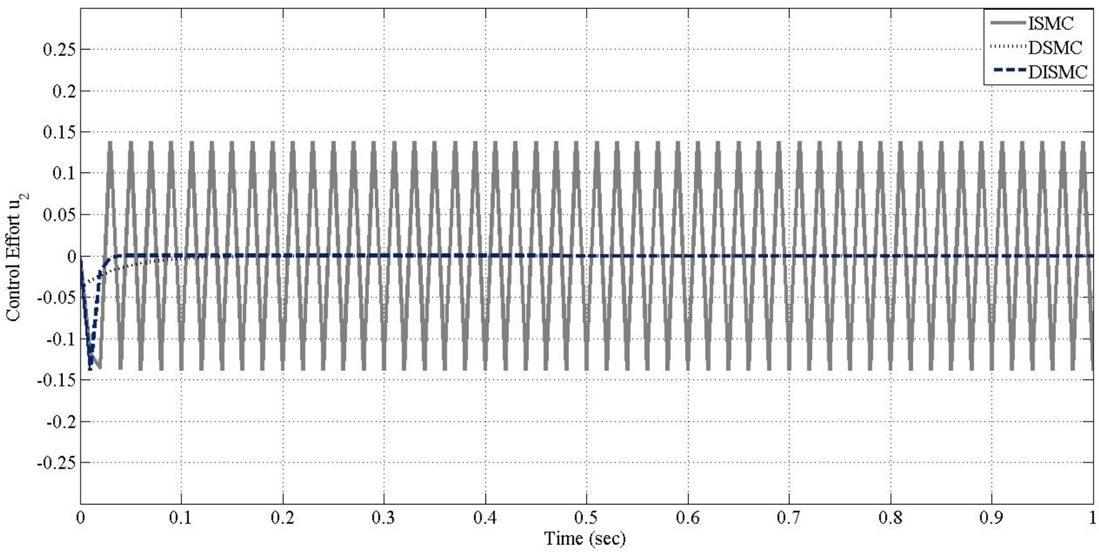


Figure 5.10: Control effort u_2 via ISMC, DSMC and DISMC in the presence of uncertainty.

5.3.4 Output Additive Uncertainty

A state dependent uncertainty

$$\zeta_i(x,t) = 3x_1 sin(\pi x_2 x_3 t), i = 1,2 \qquad \text{Eq (5.35)}$$

is introduced in both the output channels from the very beginning of the process in simulations. The outputs convergence using ISMC, DSMC and the proposed DISMC are shown in Figure 5.5 and 5.6. It is clear from both the figures that DISMC quickly steers the outputs to zero in finite settling time which is approximately equal to 0.1sec. The response time of DSMC is 0.8 and 0.1 seconds for the outputs while the outputs regulation via ISMC oscillates against the origin with considerable magnitude and high frequency. Similarly, the convergence of sliding manifolds is shown in Figure 5.7 and 5.8. It is visible that the convergence of sliding surface for DSMC and new control law DISMC are very close to each other. However, the convergence of DISMC exhibits speedy response. The sliding variable for ISMC oscillates around the origin. The control efforts of both controllers are illustrated in Figure 5.9 and 5.10. One can observe easily that the controller u_1 and u_2 via DSMC and proposed control law are chatter free while that of ISMC has chattering with significant magnitude. Thus, the proposed controller may evolve as a better controller as compared to DSMC and ISMC. The controllers gains used in this analysis are defined in Table 5.4.

TABLE 5.5: Gains of the control laws $\dot{u}_1, u_2$ used for DISMC simulations in the presence of parametric variations.

Parameters	k_{11}	k_{12}	k_{21}	K_1, K_2	K_{01}, K_{02}	c_{11}	c_{12}	c_{21}
DISMC u_1	31.6	32.6	1	3	0.001	25	1	3

5.3.5 Parametric Variations

This experiment involves the evaluation of the proposed controller under parametric variations in the Three Tank System. This system has four parameters S,c_1, c_2 and c_3 with their nominal values $0.0154, 0.0072, 0.00974$ and 0.0072, respectively.

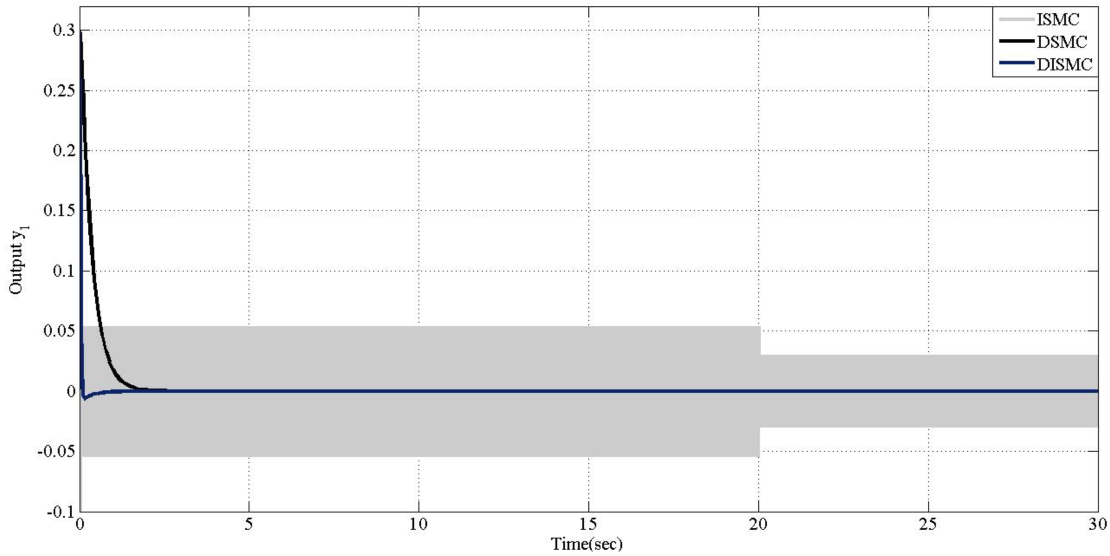


FIGURE 5.11: y_1 trajectories in the presence of parametric variation via ISMC, DSMC and DISMC.

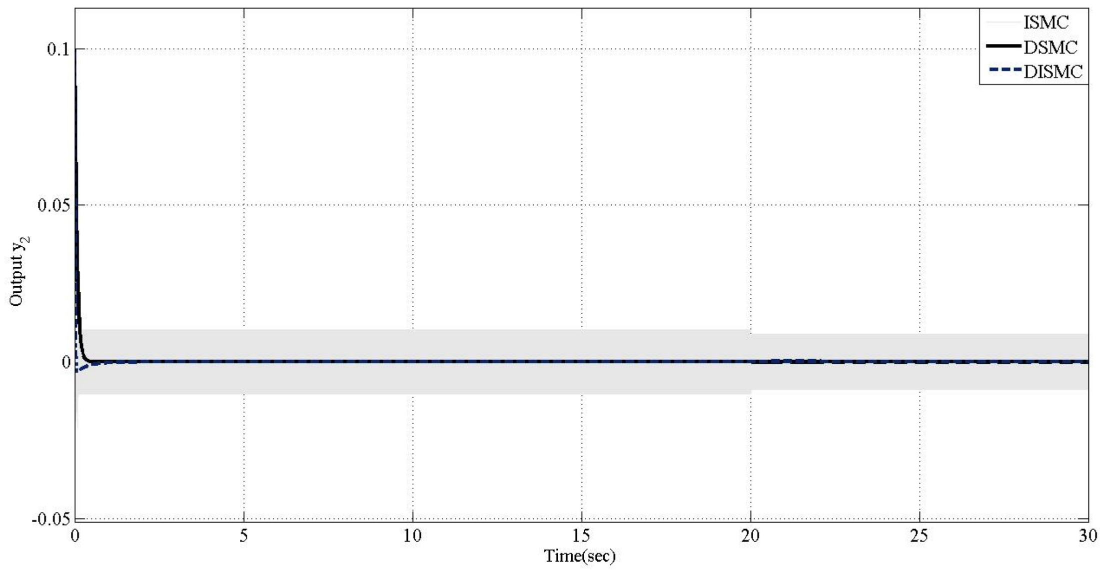


FIGURE 5.12: y_2 trajectories in the presence of parametric variation via ISMC, DSMC and DISMC.

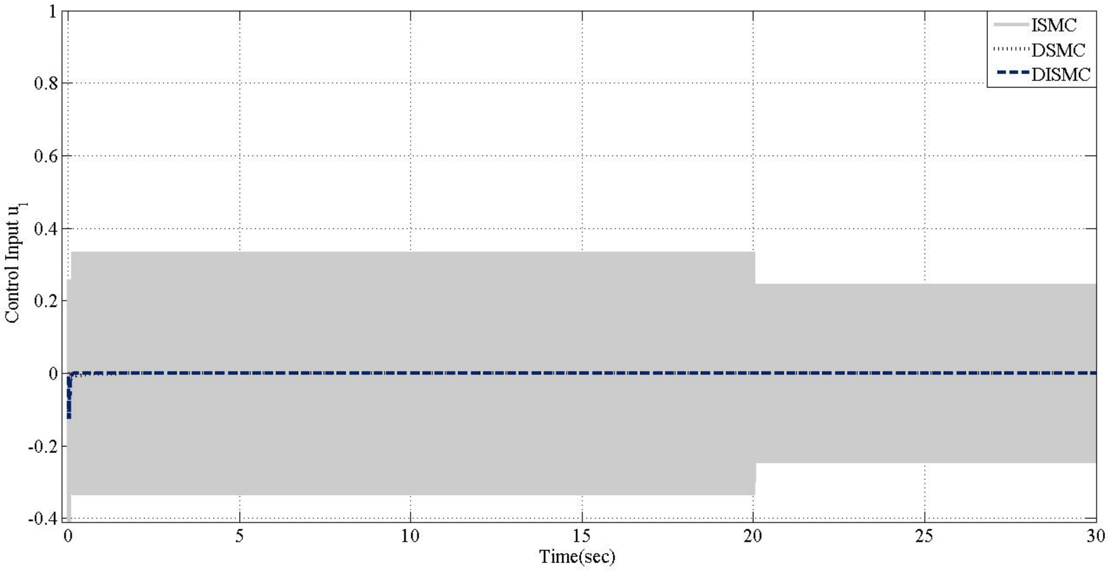


FIGURE 5.13: Control Effort u_1 via ISMC, DSMC and DISMC in the presence of parametric variation.

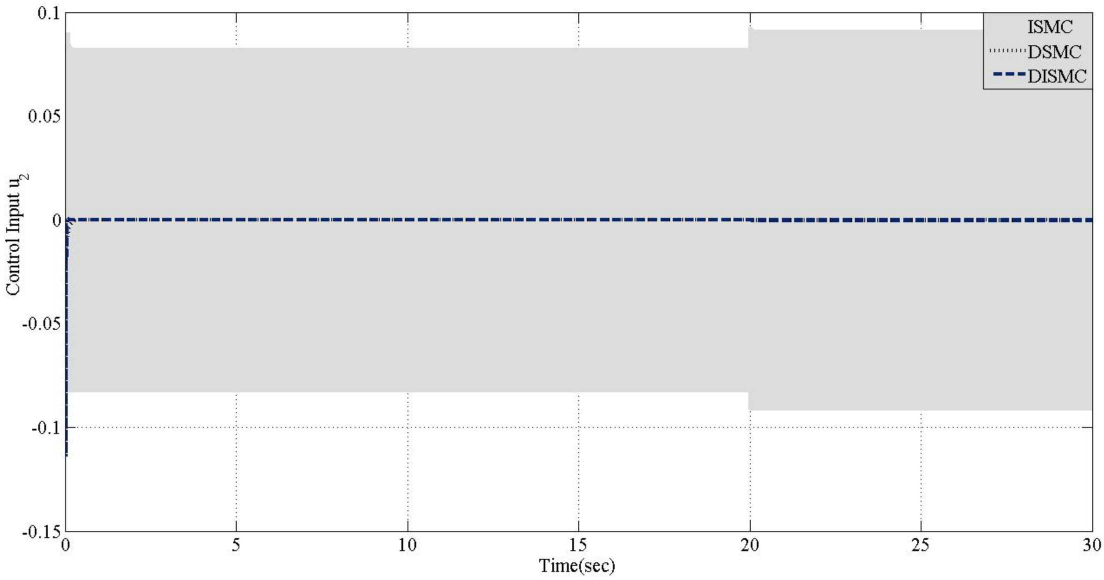


FIGURE 5.14: Control Effort u_2 via ISMC, DSMC and DISMC in the presence of parametric variation.

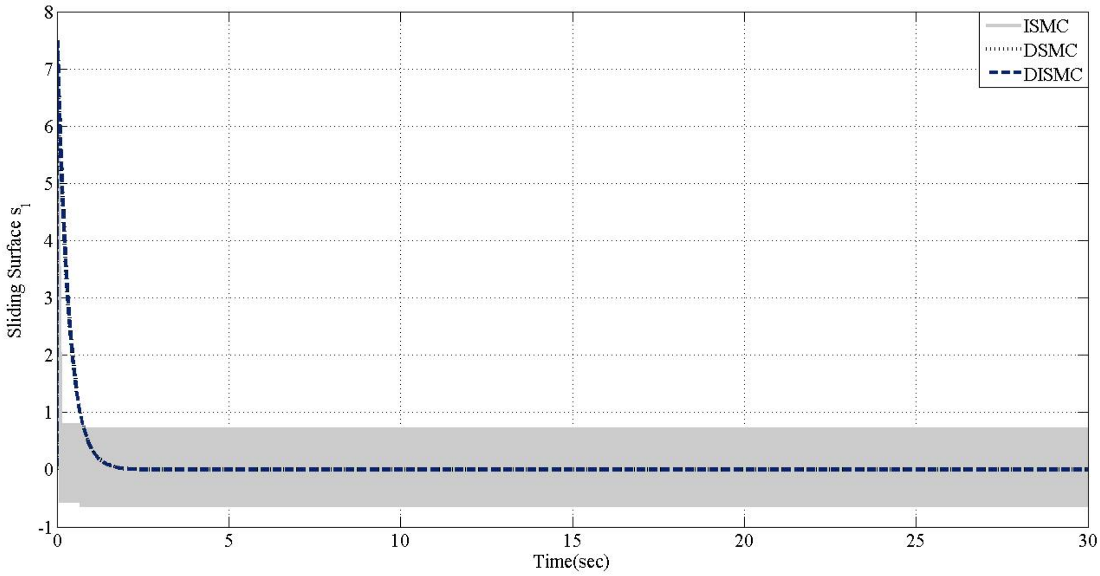


FIGURE 5.15: Sliding Surface σ_1 convergence for ISMC, DSMC and DISMC in the presence of parametric variations.

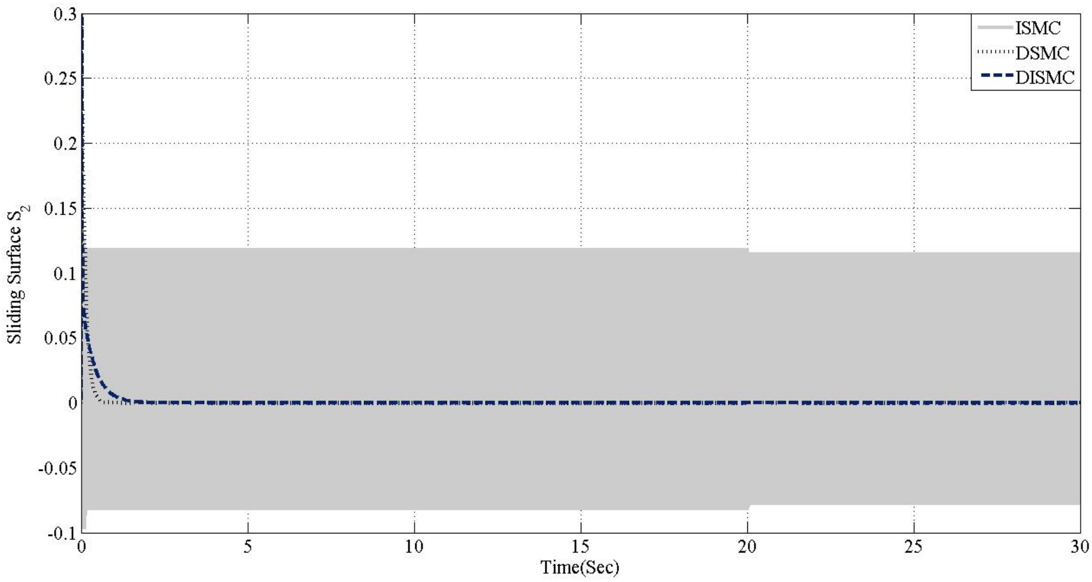


FIGURE 5.16: Sliding Surface σ_2 convergence for ISMC, DSMC and DISMC in the presence of parametric variations.

These parameters are varied with 30% increase in their nominal values. These variations are initiated in the system in steady state. The convergence of the outputs y_1 and y_2 is displayed in Figures 5.11 and 5.12. This ensures the quick and oscillation free convergence of the outputs to the origin via DISMC. On the other hand, it is obvious from these figures that outputs convergence of DSMC is not only slower than that of DISMC but is also oscillatory in behavior. In addition, the output of ISMC has fast convergence but shows oscillatory pattern with high frequency and considerable magnitude. Figures 5.13 and 5.14, confirm the chattering free nature of DISMC. It is obvious that ISMC controller experiences chattering phenomena. Similarly, the DSMC controller exhibits oscillation and a significant peak appears when the variations are introduced. In this regards, the proposed controller may be superior to the other two controllers. Same analysis can be seen in Figures 5.15 and 5.16. The convergence of sliding manifolds is displayed over there. The controller gains use in this case study are give in Table 5.5.

In the forthcoming section, the system *Eq (5.1)* is considered to be operating under a class of matched and unmatched uncertainties. The sliding mode is enforced, in finite time, against the integral manifold *Eq (5.16)* in the presence of these uncertainties.

5.4 MIMO Nonlinear System Operating under Matched and Unmatched Uncertainties

5.4.1 Problem formulation

Consider a MIMO nonlinear system represented by the state space equation analogous to that considered in [39]

$$\dot{x} = f(x,t) + g(x,t)[u(1+\delta_m) + \Delta g_m(x,t)] + f_u(x,t) \qquad Eq\ (5.36)$$

$$y = h(x) \qquad Eq\ (5.37)$$

where $x \in R^n$, $u \in R^m$, $f : R^n \times R^+ \rightarrow R^n$ and $g : R^n \times R^+ \rightarrow R^n$ and $h : R^n \rightarrow R^m$ are sufficiently smooth vector fields. The terms δ_m, $\Delta g_m(x,t)$ represents matched uncertainties which have conformable dimensions. In addition, the term $f_u(x,t)$ are unmatched uncertainties. The detailed representation of these uncertain term is given as follows. $\delta_m = [\delta_{m_1}, \delta_{m_2}, ..., \delta_{m_m}]^T$, $\Delta g_m(x,t) = [\Delta g_{m_1}(x,t), \Delta g_{m_2}(x,t), ..., \Delta g_{m_m}(x,t)]^T$ and $f_u(x,t) = [f_{u_1}(x,t), f_{u_2}(x,t), ..., f_{u_n}(x,t)]^T$. The following assumption is introduced:

Assumption 6. *The uncertainties are assumed to be continuous, norm bounded with norm bounded derivatives for all* $(x,t) \in R^n \times R^+$ *i.e,* $|\Delta g_{m_i}(x,t)| \leq \rho_{m_i}$, $|\delta_{m_i}| \leq (1 - \epsilon_{m_i})$ *and* $|f_{u_i}(x,t)| \leq \rho_{u_i}$, *where* ρ_{m_i}, ϵ_{m_i} *and* ρ_{u_i} *is some positive constants.*

In this note, the problem remain same and the problem we want to solve (Problem 1) is that of steering the vector of outputs to zero asymptotically i.e., an output regulation problem is considered here in the presence of a class of states dependent matched and unmatched uncertainties. In order to design the control law, system *Eq (5.36)* is transformed into the following form (see for instance, [35], [70], and [68])

$$\begin{aligned}
\dot{\xi}_{i1} &= \xi_{i2} + \zeta_{i1}(\hat{\xi}, \hat{u}, t) \\
\dot{\xi}_{i2} &= \xi_{i3} + \zeta_{i2}(\hat{\xi}, \hat{u}, t) \\
&\vdots \\
\dot{\xi}_{in_i} &= \varphi_i(\hat{\xi}, \hat{u}, u_k^{(\beta_k)}, t) + \gamma_i(\hat{\xi}, \hat{u}, t)[u_i^{(\beta_i)}(1+\delta_{m_i}) + \Delta G_{m_i}(\hat{\xi}, \hat{u}, t)] \\
&\quad + F_{u_i}(\hat{\xi}, \hat{u}, t)
\end{aligned} \qquad Eq\ (5.38)$$

where $\hat{\xi} = [\hat{\xi}_1, \hat{\xi}_2, ..., \hat{\xi}_n]^T$, $\hat{\xi}_i = [\dot{\xi}_i, ..., \xi_i^{(n_i-1)}]^T = [\xi_{i1}, \xi_{i2}, ..., \xi_{in})]^T$ $\hat{u} = [\hat{u}_1, \hat{u}_2, ..., \hat{u}_m]^T$, $\hat{u}_i = [\dot{u}_i, ..., u_i^{(\beta_i-1)}]^T$, for $i = 1, 2, ..., m$. The subscript n_i represents the derivative of each output such that $\sum_i^m n_i = n$. The term $\varphi_i(\hat{\xi}, \hat{u}, u_k^{(\beta_k)}, t)$ represents the nominal part of the system where as $\zeta_{ij}(\hat{\xi}, \hat{u}, t)$ and $F_{u_i}(\hat{\xi}, \hat{u}, t)$ refers to the uncertainties. The representation in *Eq (5.38)* is analogous to the so-called Local

Generalized Controllable Canonical (LGCC) form [65] in the sense that it differs from the basic LGCC form since it is affected by matched and unmatched uncertainties. With reference to system *Eq (5.38)*, the forthcoming assumption (which is an alternative form of Assumption 6) is introduced:

Assumption 7. *Assume that* $|\varphi_i(\hat{\xi}, \hat{u}, u_k^{(\beta_k)}, t)| \leq C_i$, $|\gamma_i(\hat{\xi}, \hat{u}, t)| \leq K_{M_i}$, $|\Delta G_{m_i}(x,t)| \leq B_i$, $|F_{u_i}(x,t)| \leq \lambda_i$, $|\zeta_{ij}(\hat{\xi}, \hat{u}, t)| \leq \mu_i$ *for* $j = 1, 2, ..., n_i - 1$, *where* C_i, K_{M_i}, B_i, λ_i, μ_i *are positive constants. Furthermore, assume that* $\sum_{j=1}^{(n_i-1)} \zeta_{ij}(\hat{\xi}, \hat{u}, t) + F_{u_i}(\hat{\xi}, \hat{u}, t) \equiv \Theta_i(\hat{\xi}, \hat{u}, t)$ *and is bounded by the positive constants* τ_i *i.e.,* $|\Theta_i(\hat{\xi}, \hat{u}, t)| \leq \tau_i$.

The following nominal system corresponding to *Eq (5.38)* can be obtained when $\delta_{m_i} = 0$, $\Delta G_{m_i}(\hat{\xi}, \hat{u}, t) = 0$, $\zeta_{ij}(\hat{\xi}, \hat{u}, t) = 0$ and $\Theta_i(\hat{\xi}, \hat{u}, t) = 0$.

$$\begin{aligned} \dot{\xi}_{i1} &= \xi_{i2} \\ \dot{\xi}_{i2} &= \xi_{i3} \\ &\vdots \\ \dot{\xi}_{in_i} &= \varphi_i(\hat{\xi}, \hat{u}, t) + \gamma_i(\hat{\xi}, \hat{u}, t) u_i^{(\beta_i)} \end{aligned} \qquad Eq\ (5.39)$$

Now, (Problem 1) can be reformulated with reference to system *Eq (5.38)* under Assumption 7, and to the nominal system in *Eq (5.39)*. Therefore, the new control problem (Problem 2) is to regulate to zero asymptotically the vector of outputs $\hat{\xi} = [\hat{\xi}_1, \hat{\xi}_2, ..., \hat{\xi}_n]^T$ in the presence of matched and unmatched uncertainties. In other words, a regulation problem is considered here. The solution to (Problem 2) is a clear solution to (Problem 1) since $y = [\xi_{11}, \xi_{21}, ..., \xi_{n1}]^T$.

The control law proposed in the previous study of this chapter make use of the strong reachability condition [16]. In the forthcoming work we use the famous reachability condition [33]

$$\dot{\sigma}_i = -K_i sign(\sigma_i) \qquad Eq\ (5.40)$$

Therefore, the discontinuous control component *Eq (5.21)* will be replaced with the following discontinuous control component.

$$u_{1i}^{(\beta_i)} = -(1/\gamma_i(\hat{\xi}, \hat{u}, t))[\varphi_i(\hat{\xi}, \hat{u}, u_k^{(\beta_k)}, t) + (\gamma_i(\hat{\xi}, \hat{u}, t) - 1)u_{0i}^{(\beta_i)}$$

$$+ K_i sign(\sigma_i)] \qquad \text{Eq (5.41)}$$

This control law enforces sliding mode along the sliding manifold defined $\sigma_i(\hat{\xi}_i) = 0$ in *Eq (5.16)*. The constants K_i can be selected according to the subsequent stability analysis. Thus, the final control law becomes

$$u_i^{(\beta_i)} = -K_i^T \hat{\xi}_i - (1/\gamma_i(\hat{\xi}, \hat{u}, t))[\varphi_i(\hat{\xi}, \hat{u}, u_k^{(\beta_k)}, t) + (\gamma_i(\hat{\xi}, \hat{u}, t) - 1)u_{0i}^{(\beta_i)}$$

$$+ K_i sign(\sigma_i)] \qquad \text{Eq (5.42)}$$

5.5 Stability Analysis

In this section, the proposed control law when applied to the uncertain nonlinear system in question is theoretically analyzed. First the case in which only matched uncertainties are present will be discussed, and then, the more general case of matched and unmatched uncertainties will be considered.

5.5.1 The System Operating Under Matched Uncertainties

Now we assume that the system operates only under matched uncertainties. Thus, system *Eq (5.38)* with only matched uncertainties becomes

$$\dot{\xi}_{i1} = \xi_{i2}$$

$$\dot{\xi}_{i2} = \xi_{i3}$$

$$\vdots \qquad \text{Eq (5.43)}$$

$$\dot{\xi}_{in_i} = \varphi_i(\hat{\xi}, \hat{u}, u_k^{(\beta_k)}, t) + \gamma_i(\hat{\xi}, \hat{u}, t)[u_i^{(\beta_i)}(1 + \delta_{m_i}) + \Delta G_{m_i}(\hat{\xi}, \hat{u}, t)]$$

To show that this system is stabilized, in finite time, in the presence of matched uncertainties, the following theorem can be stated.

Theorem 5.4. *Consider that Assumptions 5 and 7 are satisfied. The sliding surface is chosen as* $\sigma_i(\hat{\xi}) = 0$*, where* σ_i *is defined in Eq (5.16), and the control law is selected according to Eq (5.42). If the gain is chosen according to the following condition Eq (5.44)*

$$K_i \geq \frac{1}{(2 - \epsilon_{m_i})}[(1 - \epsilon_{m_i})|u_{0i}^{(\beta_i)}| + (1 - \epsilon_{m_i})C_i + K_{M_i}B_i + \eta_{1i}] \qquad \text{Eq (5.44)}$$

where η_{1i} *is some positive constant, then, the finite time enforcement of a sliding mode on* $\sigma_i(\hat{\xi}) = 0$ *is guaranteed in the presence of matched uncertainties.*

Proof. To prove that the sliding mode can be enforced in finite time, differentiating *Eq (5.16)* along the dynamics of *Eq (5.43)*, and then substituting *Eq (5.18)* and *Eq (5.42)*, one has

$$\dot{\sigma}_i = -K_i sign(\sigma_i) + \delta_{m_i}[(u_{0i}^{(\beta_i)} - \varphi_i(\hat{\xi}, \hat{u}, u_k^{(\beta_k)}, t) - K_i sign(\sigma_i))]$$

$$+ \gamma_i(\hat{\xi}, \hat{u}, t)\Delta G_{m_i}(\hat{\xi}, \hat{u}, t) \qquad \text{Eq (5.45)}$$

Now, by considering as a Lyapunov candidate function $v_i = (1/2)(\sigma_i)^2$, the time derivative of this function along *Eq (5.45)* becomes

$$\dot{v}_i \leq |\sigma_i|[-K_i(1 + |\delta_{m_i}|) + |\delta_{m_i}||u_{0i}^{(\beta_i)}| + |\varphi_i(\hat{\xi}, \hat{u}, u_k^{(\beta_k)}, t)|$$

$$+ |\gamma_i(\hat{\xi}, \hat{u}, t)\Delta G_{m_i}(\hat{\xi}, \hat{u}, t)|] \qquad \text{Eq (5.46)}$$

In view of Assumption 7, the above expression in *Eq (5.46)* can be written as follows

$$\dot{v}_i \leq |\sigma_i|[-K_i(2 - \epsilon_{m_i}) + (1 - \epsilon_{m_i})|u_{0i}^{(\beta_i)}| + (1 - \epsilon_{m_i})C_i + K_{M_i}B_i]$$

or

$$\dot{v}_i \leq -\eta_{1i}|\sigma_i| \qquad \textit{Eq (5.47)}$$

Provided that

$$K_i \geq \frac{1}{(2-\epsilon_{m_i})}[(1-\epsilon_{m_i})|u_{0i}^{(\beta_i)}| + (1-\epsilon_{m_i})C_i + K_{M_i}B_i + \eta_{1i}] \qquad \textit{Eq (5.48)}$$

as in *Eq (5.44)*. Note that *Eq (5.47)* can also be written as

$$\dot{v}_i + \sqrt{2}\eta_{1i}\sqrt{v_i} < 0 \qquad \textit{Eq (5.49)}$$

This implies that $\sigma_i(\hat{\xi}) = 0$ is reached in finite time t_{s_i} (see [46]), such that

$$t_{s_i} \leq \sqrt{2}\eta_{1i}^{-1}\sqrt{v_i(\sigma_i(0))} \qquad \textit{Eq (5.50)}$$

which completes the proof.

Corollary 5.5. *The dynamics of the system Eq (5.43) in the absence of unmatched uncertainties, with control law Eq (5.42) and integral manifold Eq (5.16), in sliding mode is governed by the linear control law Eq (5.14).*

Proof. To prove the above claim, differentiating *Eq (5.16)* along *Eq (5.38)*, and then substituting *Eq (5.18)*, one has the following

$$\dot{\sigma}_i = \varphi_i(\hat{\xi}, \hat{u}, u_k^{(\beta_k)}, t) + (\gamma_i(\hat{\xi}, \hat{u}, t) - 1)u_{0i}^{(\beta_i)} + \gamma_i(\hat{\xi}, \hat{u}, t)u_{1i}^{(\beta_i)} + \gamma_i(\hat{\xi}, \hat{u}, t)[u_i^{(\beta_i)}(1 + \delta_{m_i})$$

$$+ \Delta G_{m_i}(x,t)] + \Theta_i(\hat{\xi}, \hat{u}, t) \qquad \textit{Eq (5.51)}$$

Since we are taking only the matched uncertainties into account, therefore, substituting $\Theta_i(\hat{\xi}, \hat{u}, t) = 0$, one has $\dot{\sigma}_i = \varphi_i(\hat{\xi}, \hat{u}, u_k^{(\beta_k)}, t) + (\gamma_i(\hat{\xi}, \hat{u}, t) - 1)u_{0i}^{(\beta_i)} + \gamma_i(\hat{\xi}, \hat{u}, t)u_{1i}^{(\beta_i)}$

$$+ \gamma_i(\hat{\xi}, \hat{u}, t)[u_i^{(\beta_i)}(1 + \delta_{m_i}) + \Delta G_{m_i}(x,t)] \qquad \textit{Eq (5.52)}$$

or

$$\dot{\sigma}_i = \varphi_i(\hat{\xi}, \hat{u}, u_k^{(\beta_k)}, t) + (1 + \delta_{m_i})\gamma_i(\hat{\xi}, \hat{u}, t){u_i}^{(\beta_i)} - {u_{0i}}^{(\beta_i)}$$

$$+ \gamma_i(\hat{\xi}, \hat{u}, t)\Delta G_{m_i}(x,t) \qquad \textit{Eq (5.53)}$$

Now, posing $\dot{\sigma}_i = 0$, and solving with respect to the control variable $u_i^{(\beta_i)}$, one obtains the so-called equivalent control [34] as

$$u_{eq}^{(\beta_i)} = \Big(1/(1+\delta_{m_i})\gamma_i(\hat{\xi}, \hat{u}, t)\Big)\,[-\varphi_i(\hat{\xi}, \hat{u}, u_k^{(\beta_k)}, t) + u_{0i}^{(\beta_i)} + \gamma_i(\hat{\xi}, \hat{u}, t)\Delta G_{m_i}(x, t)] \qquad Eq\ (5.54)$$

Now, using *Eq (5.54)* in *Eq (5.43)*, one has

$$\dot{\hat{\xi}}_{i,s} = A_i\hat{\xi}_{i,s} + B_i u_{0i}{}^{(\beta_i)} \qquad Eq\ (5.55)$$

where A_i and B_i has the form discussed in Section 5.2 and $\hat{\xi}_{i,s}$ is the state vector of the system *Eq (5.43)*. Thus, it is proved that the system in sliding mode operates under the continuous control law and the eigenvalues of the controlled transformed system in sliding mode are those of $A_i - B_i K_i^T$.

5.5.2 The System Operating Under both matched and unmatched Uncertainties

In this subsection, it is now assumed that the considered system operates under both matched and unmatched uncertainties and the control objective is to regulate the output of the system in the presence of these uncertainties. To prove that the proposed control law is capable of compensating for these uncertain terms, the following theorem can be stated.

Theorem 5.6. *Consider that Assumptions 5 and 7 are satisfied. The sliding surface is chosen as $\sigma_i(\hat{\xi}) = 0$, where σ_i is defined in Eq (5.16), and the control law is selected according to Eq (5.42). If the gain is chosen according to the following condition Eq (5.56)*

$$K_i \geq \frac{1}{(2-\epsilon_{m_i})}[(1-\epsilon_{m_i})|u_{0i}^{(\beta_i)}| + (1-\epsilon_{m_i})C_i + K_{M_i}B_i + \eta_{2i} + \tau_i] \qquad Eq\ (5.56)$$

where η_{2i} is some positive constant, then, the finite time enforcement of a sliding mode on $\sigma_i(\hat{\xi}) = 0$ is guaranteed in the presence of both matched and unmatched uncertainties.

Proof. To prove that the sliding mode can be enforced in finite time, the time derivative of the Lyapunov candidate function $v_i = (1/2)(\sigma_i)^2$, along *Eq (5.51)* becomes as follows

$$v_i \leq |\sigma_i|[-K_i(1+|\delta_{mi}|)+|\delta_{m_i}||u_{0i}^{(\beta_i)}|+|\varphi_i(\hat{\xi},\hat{u},u_k^{(\beta_k)},t)| + |\gamma_i(\hat{\xi},\hat{u},t)\Delta G_{m_i}(\hat{\xi},\hat{u},t)|+|\Theta_i(\hat{\xi},\hat{u},t)|] \qquad Eq\ (5.57)$$

In view of Assumption 7, the above expression can be written as

$$\dot{v}_i \leq |\sigma_i|[-K_i(2-\epsilon_{m_i})+(1-\epsilon_{m_i})|u_{0i}^{(\beta_i)}|+(1-\epsilon_{m_i})C_i+K_{M_i}B_i+\tau_i]$$

or

$$\dot{v}_i \leq -\eta_{2i}|\sigma_i| \qquad Eq\ (5.58)$$

Provided that

$$K_i \geq \frac{1}{(2-\epsilon_{m_i})}[(1-\epsilon_{m_i})|u_{0i}^{(\beta_i)}|+(1-\epsilon_{m_i})C_i+K_{M_i}B_i+\eta_{2i}+\tau_i] \qquad Eq\ (5.59)$$

The expression in *Eq (5.59)* can be placed in the same format like that of *Eq (5.49)*. Note that the finite time t_{s_i} in this case is given by the formula in *Eq (5.50)* with η_{2i} instead of η_{1i}. Thus it is confirmed that, when the gain of the discontinuous component of the control law *Eq (5.42)* is selected according to *Eq (5.56)*, the finite time enforcement of the sliding mode is guaranteed in the presence of matched and unmatched uncertainties, which proves the theorem.

Corollary 5.7. *The dynamics of the system Eq (5.38), with control law Eq (5.42) and integral sliding manifold $\sigma_i = 0$, with $\sigma_i(\hat{\xi})$ defined in Eq (5.16), in sliding mode is governed by the linear control law Eq (5.14).*

Proof. The proof can be performed by following the same procedure as in the proof of Corollary 1, with the only difference that in this case the equivalent control is equal to $u_{eq}^{(\beta_i)} = \left(1/(1+\delta_{im})\gamma_i(\hat{\xi},\hat{u},t)\right)[-\varphi_i(\hat{\xi},\hat{u},u_k^{(\beta_k)},t)+u_{0i}^{(\beta_i)}$

$$+\gamma_i(\hat{\xi},\hat{u},t)\Delta G_{im}(x,t)+\Theta_i(\hat{\xi},\hat{u},t)] \qquad Eq\ (5.60)$$

5.6 Illustrative Example

Consider a MIMO nonlinear system represented by the following state space equations [71]

$$\dot{x}_1 = x_2 + f_1(x,t)$$

$$\dot{x}_2 = qx_1 + x_3^2 + x_1x_4cos(x_3) + [u_1(1+\delta_{m_1}) + \Delta G_{m_1}(x,t)] + f_2(x,t)$$

$$\dot{x}_3 = x_4 + f_3(x,t) \qquad \textit{Eq (5.61)}$$

$$\dot{x}_4 = wx_1^3 - x_1cos(x_3) + [u_2(1+\delta_{m_2}) + \Delta G_{m_2}(x,t)] + f_4(x,t)$$

where x_1, x_2, x_3 and x_4 are states and u_1 and u_2 are the inputs to the nonlinear system. q and w are the parameters with their nominal values 3 and 1, respectively. The outputs of interest are $y_1 = x_2$ and $y_2 = x_4$. In this study, the objective is to regulate the outputs from some initial position to the desired equilibrium point (origin). The relative degree of the system with respect to both the output functions is 1. The terms δ_{m_i}, and $\Delta G_{m_i}(x,t)$, $i = 1,2$, are matched uncertainties and $f_j(x,t)$, $j = 1,2,3,4$ are the components of the unmatched uncertainty which satisfy Assumptions 5, 6 and 7 and these terms contribute to the system uncertainty with the following mathematical expressions.

$$f_1(x,t) = 0.2x_3x_4 + 0.1x_3sin(x_2^2) + 10x_3cos(x_1) + 0.1$$

$$f_2(x,t) = 0.15x_4x_2 + 10x_3cos(x_1) + 0.1$$

$$f_3(x,t) = 0.5x_1x_2 + 0.15x_3x_4 + 10x_1cos(x_3) + 0.1$$

$$f_4(x,t) = .1(x_4x_2 + 10x_1cos(x_3) + 0.1$$

$$\Delta G_{m_1}(x,t) = 0.2x_3^2x_4$$

$$\Delta G_{m_1}(x,t) = 0.5x_1x_2$$

$$\delta_{m_1} = 0.3x_3x_4$$

$$\delta_{m_2} = 0.31x_1x_2$$

The nominal system in LGCC form for the above system *Eq (5.61)* becomes

$$\dot{\xi}_{11} = \xi_{12}$$

$$\dot{\xi}_{12} = \varphi_1(\hat{\xi}_1, \hat{\xi}_2, \hat{u}_2) + \dot{u}_1$$

$$\dot{\xi}_{21} = \xi_{22} \qquad \textit{Eq (5.62)}$$

$$\dot{\xi}_{22} = \varphi_2(\hat{\xi}_1, \hat{\xi}_2) + \dot{u}_2$$

where

$$\varphi_1(\hat{\xi}_1, \hat{\xi}_2, \hat{u}_2) = [qx_2 + 2x_3x_4 + x_2x_4Cos(x_3)$$

$$+ x_1(wx_1^3 - x_1Cos(x_3) + u_2)cos(x_3) - x_1x_4^2sin(x_3)] \qquad \textit{Eq (5.63)}$$

$$\varphi_2(\hat{\xi}_1, \hat{\xi}_2) = 3x_1^2x_2 - x_4Cos(x_1) - x_2x_3sin(x_1)$$

$\hat{\xi}_1 = [\xi_{11}, \xi_{12}]^T$, $\hat{\xi}_2 = [\xi_{21}, \xi_{22}]^T$, $\hat{u}_1 = u_1$, $\hat{u}_2 = u_2$ and $\hat{\xi} = [\hat{\xi}_1, \hat{\xi}_2]^T$.

$$\gamma_1(\hat{\xi}) = 1$$

$$\gamma_2(\hat{\xi}) = 1$$

$\xi_{11} = x_2$, $\xi_{12} = qx_1 + x_3^2 + x_1x_4cos(x_3) + u_1$, $\xi_{21} = x_4$ and $\xi_{22} = wx_1^3 - x_1cos(x_3) + u_2$. Now, the corresponding linear systems becomes

$$\dot{\hat{\xi}}_i = A_i\hat{\xi}_i + B_i\dot{u}_{0i}, i = 1, 2 \qquad \textit{Eq (5.64)}$$

where each $\hat{\xi}_i = [\xi_{i1}, \xi_{i2}]^T$ is the state vector of the outputs and its derivatives. $A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for $i = 1, 2$. The continuous components of the control law becomes

$$\dot{u}_{0i} = k_{i1}\xi_{i1} + k_{i2}\xi_{i2}$$

Note that, the continuous control components are designed via pole placement. The design of the discontinuous components is carried by first designing the sliding surfaces as follows

$$\sigma_1 = c_{11}\xi_{11} + \xi_{12} + z_1$$

$$\sigma_2 = c_{21}\xi_{21} + \xi_{22} + z_2$$

$$\dot{z}_1 = -\dot{u}_{01} - c_{11}\xi_{12}$$

$$\dot{z}_2 = -\dot{u}_{02} - c_{21}\xi_{21}$$

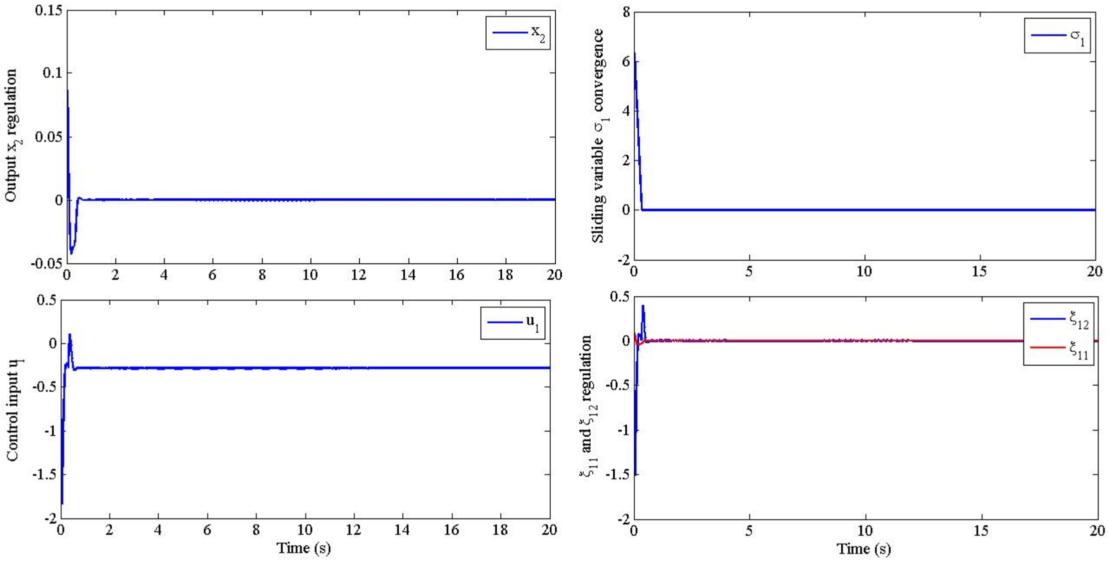


FIGURE 5.17: Output regulation, control effort, sliding variable convergence and $\hat{\xi}_1$ regulation in the presence of matched uncertainty

and the expression of both the discontinuous controller takes the form

$$\dot{u}_{11} = -\varphi_1(\hat{\xi}_1, \hat{\xi}_2, \hat{u}_2) - K_1 sign(\sigma_1)$$

$$\dot{u}_{12} = -\varphi_2(\hat{\xi}_1, \hat{\xi}_2) - K_2 sign(\sigma_2)$$

Therefore, the final form for the control laws can be obtained by inserting the values of the respective continuous and discontinuous components in *Eq (5.10)*. This completes the controller design for the prescribed system. In the forthcoming work, the above system is simulated in the presence of matched and unmatched uncertainties.

5.6.0.1 System operated with matched uncertainties

In this study, the system with matched uncertainties (i.e., $f_i(x,t) = 0$ for $i = 1, 2, 3, 4$) is simulated to confirm the aforementioned claim of the compensation of uncertain terms. The results are reported in Figures 5.18 and 5.19. In these Figures, it can be seen that the output system with state vector is regulated in the presence of uncertainties. It is noticeable that the proposed methodology provides a satisfactory regulation of the system output via a continuous control law.

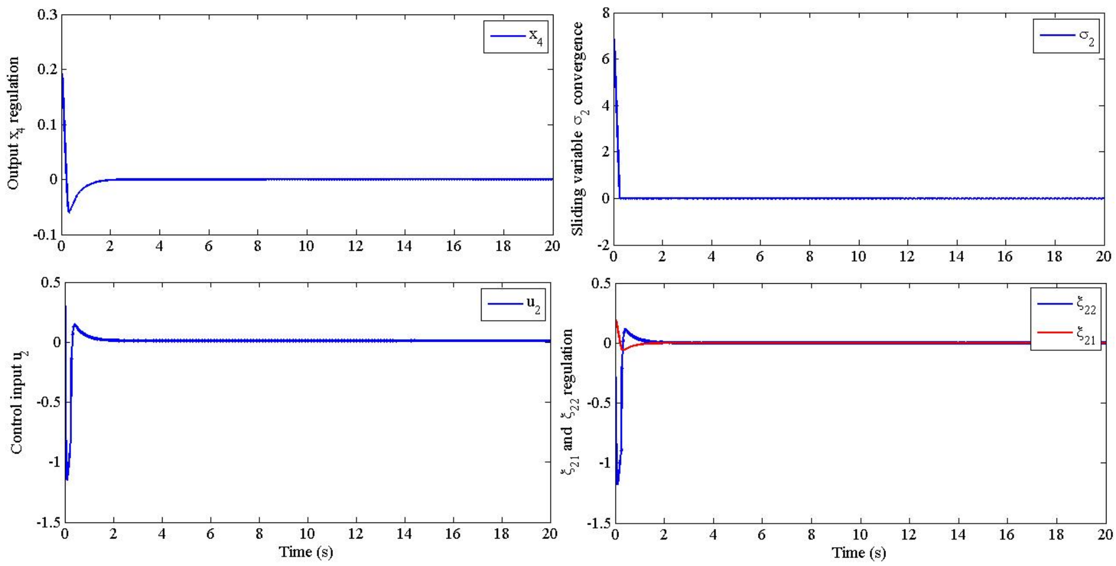


FIGURE 5.18: Output regulation, control effort, sliding variable convergence and $\hat{\xi}_2$ regulation in the presence of matched uncertainty

TABLE 5.6: Gains of the Controllers

c_{11}	c_{21}	k_{11}	k_{12}	k_{21}	k_{22}	K_1	K_2
55	34	-570	-32	-70	-32	20	30

5.6.0.2 System operated under both matched and unmatched uncertainties

In this subsection, the test with both matched and unmatched uncertainty is performed. The results with the proposed control law are depicted in Figures 5.20 and 5.21. These simulation results confirm the robust and chattering free nature of the proposed controller as well as it capability of efficiently solving the regulation problem even in this particularly critical case.

Note that the controller gains and the controllers parameters in both the tests are listed in Table 1.

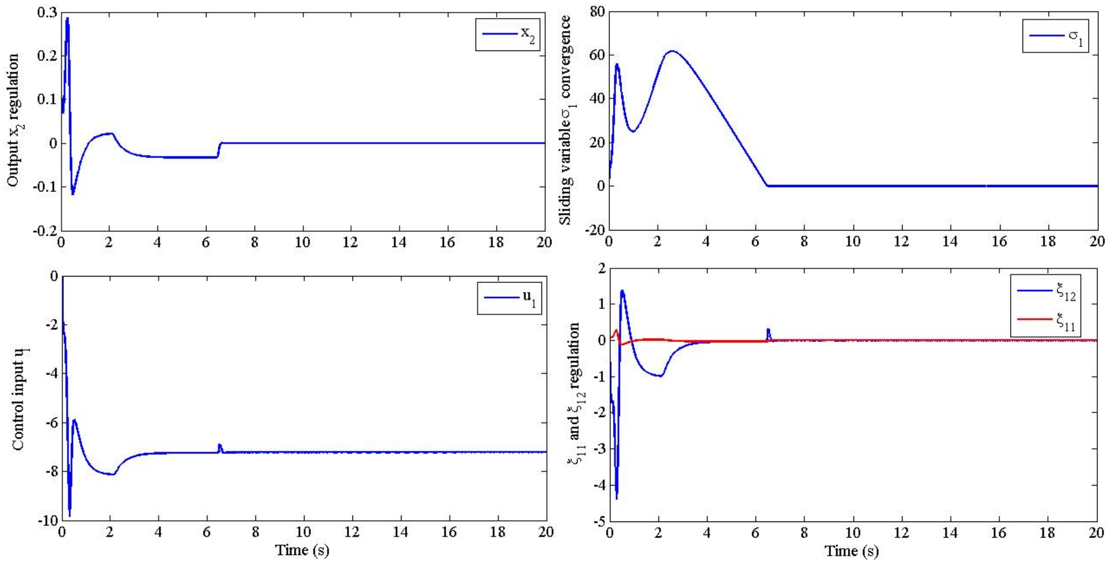


FIGURE 5.19: Output regulation, control effort, sliding variable convergence and $\hat{\xi}_1$ regulation in the presence of matched and unmatched uncertainty

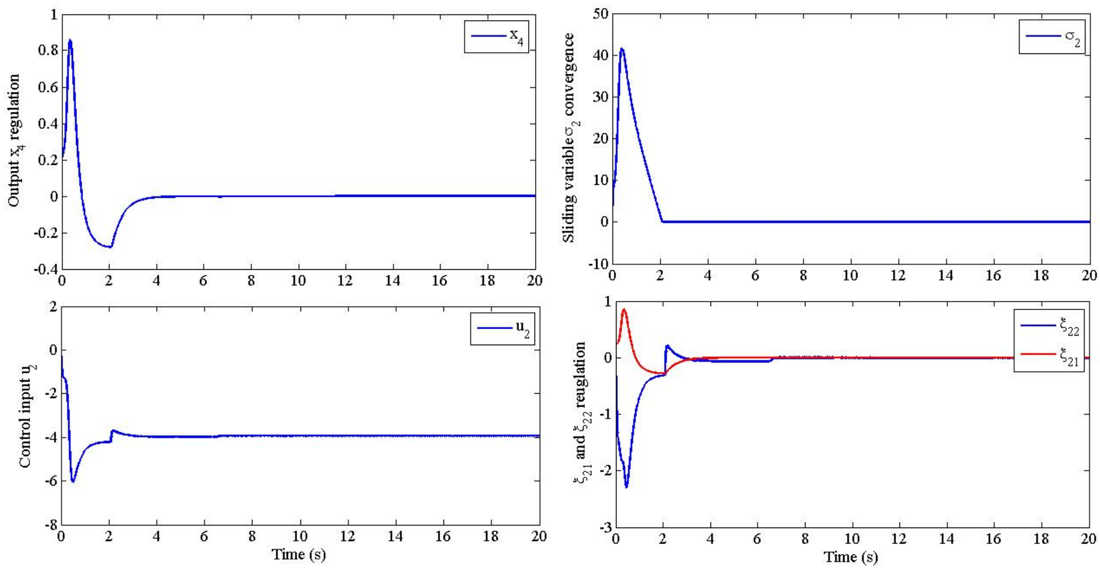


FIGURE 5.20: Output regulation, control effort, sliding variable convergence and $\hat{\xi}_2$ regulation in the presence of matched and unmatched uncertainty

5.7 Summary

In this work, the control methodology proposed in the previous chapter is extended for a class of MIMO uncertain nonlinear systems. The properties and benefits, being claimed already, are satisfied for this MIMO extension. Therefore, it is claimed that the new designed technique captured the good features of ISMC and DSMC. These claim are verified by few examples. The first one is adapted from that of [16] and the results of this newly developed control scheme is compared with the standard results of literature of DSMC. The other examples are simulated and compared with DSMC and ISMC results. Based on the simulation results it can also be claimed that the proposed DISMC outshines DSMC and ISMC controls in maximum aspects. Furthermore, this chapter includes the presentation of the rejection of the matched and unmatched uncertainties with the use of an output feedback dynamic integral sliding mode control ([21] and [22]). The gains of the discontinuous control law MIMO can be selected with the satisfaction of some inequalities. An illustrative example is taken into account to prove that the unmatched uncertainties are compensated effectively.

In the next chapter, the conclusion of the contributing chapters is given and some suggestion about the future work are given.

Chapter 6

CONCLUSION AND FUTURE WORK

6.1 Conclusions

The switching control design scheme, based on its interesting and attractive features, provided very promising control of real life systems since its introduction. However, the dangerous chattering phenomena, ever wanted robustness and performance were the basic issues which were greatly emphasized for the last forty years. A number of devoted researcher defined interesting approaches to have a solutions for the above problems. This manuscript introduced some of the well known schemes of the founders of the theory of sliding mode control. These includes the Boundary Layer approach, observer based approaches, HOSM approaches Dynamic Sliding Mode approaches for chattering attenuation. The robustness enhancement approaches once again include the Dynamic Sliding Mode approach, Differentiator bases HOSMC, ISMC, H_∞ Control in combination with SMC schemes, etc. It is obvious that the main focused issues were chattering alleviation and robustness enhancement. The existing literature also includes a number of approaches which were used to overcome the chattering reduction, robustness enhancement and performance improvement, simultaneously. The presented research work introduced an improved variant of sliding mode control which synthesized Dynamic Sliding Mode Control and Integral Sliding Mode Control techniques. This proposed control technique, being named as Dynamic Integral Sliding Mode Control (DISMC), differs from the traditional DSMC approach and integral sliding mode control in some aspects. The usual DSMC approach design is either based on direct sliding surface or indirect sliding surface with a dynamic discontinuous control law. The control law being designed via DSMC approach has very fine results with reduced chattering with robust acceptable performance. On the other hand, Integral Sliding Mode control establishes sliding mode without reaching phase which enhances the robustness against uncertainties along with acceptable performance. In order to have a good understanding of the proposed control methodology, the general design frame works of Dynamic Sliding Mode and Integral Sliding Mode are presented. The results of the new work are compared with some standard results of [15], [16] and [59] for authentication. The referred papers have used semi high gain differentiator observer. Thus, a very short overview of semi high observer is also presented. Having established the background, a detailed design procedure of the proposed control scheme for Single Input Single Output nonlinear systems is presented. This control design enjoys the good features of both the DSMC and ISMC. By good features we mean that this method make use of integral manifold approach being designed in the form of the phase variables. The integral surface comprises of two terms mainly. The first one is the usual sliding surface which always appears in the form of Hurwitz polynomial and the second part is called an integral term which mainly provides extra dynamics and helps in reaching phase elimination and, consequently, sliding mode occurs from the very start of the process. This developed scheme provides us a dynamic control law which comprises of two parts. The first part is a continuous dynamic linear control law which emphasizes on the performance improvement and the second part is a discontinuous dynamic controller which is utilized in the rejection of uncertainties and sliding mode establishment. A general nonlinear system is studied and a control law is designed with proved convergence condition. The claim is verified with two illustrative examples. The proposed design frame work is extended to Multi Input Multi Output uncertain nonlinear systems and very promising results are established for some counter examples. The robustness of the newly developed control law is also tested in the studies of SISO and MIMO nonlinear systems operating under matched and mismatched uncertainties. Some conditions are derived while using a Lyapunove candidate functions. When the gains of the control law, being designed via the new developed control methodology, are selected according the derived conditions then the uncertainties are effectively rejected. Illustrative examples for both SISO and MIMO case are presented to ensure the uncertainties rejection. The major contributions to the presented work are listed in the bulleted form in the beginning of the manuscript.

6.2 Recommended Future Work

The proposed work can be extended in theoretical as well as in application perspectives. In theoretical point of view this can be extended to the following research contributions

- **Control of under actuated nonlinear systems** is one of the major area of research. Systems being listed in the category of underactuated systems deals with more system configuration than the control inputs.Since, in this monograph, the control scheme is designed for only square plants so it can be extended for this class of nonlinear systems.

- **Disturbance estimator and parameter estimator** can be designed which can be used in a number of applications where high robust performance is required.

- **To design an adaptation** based DISMC. This approach will result in adaptive gains development and can be applied to system where adaptive approaches becomes suitable candidates.

- **Control of Non minimum Phase Nonlinear System:** The developed control methodology can also be employed to the control of those nonlinear systems which are non minimum phase system. The non-minimum phase system of both SISO and MIMO can be easily carried out.

- **Control of Nonlinear Minimum Phase Systems with Time varying disturbance** (deterministic in nature) will be developed for the nonlinear systems. These time varying disturbances may be matched and unmatched in nature. The control design may be developed with use of H_{inf} control technique.

- **Combined Stability of the Controller and Differentiator** will be carried out. The Separation Principle will be elaborated which will provide the facility to construct the controller and differentiator separately.

From application point of view the developed scheme finds interesting applications in the following fields

- The **Control of Electro mechanical systems** like DC motors, synchronous motors, switch reluctance motors, can be facilitate via this new developed control algorithm in term of fast convergence to the desired values of the outputs with desired robustness and minimized chattering.

- **Electrochemical systems** like fuel cell is an environment friendly system which provides electricity with such byproducts which are not harmful to the surrounding. The input flow rates and output control in such system is necessary for a control engineer. The developed methodology can provide robust performance in the presence of parametric variations.

- **Robotic systems**

- The tracking control in **UAV's**

REFERENCES

[1] J.Y. Hung, W. Gao, and J.C. Hung. Variable structure control: a survey. *IEEE Transactions on, Industrial Electronics*, 40(1):2 –22, feb 1993.

[2] V. I. Utkin. Variable structure systems: Present and future. *Automation and Remote Control*, 44(1):1105 –1120, feb 1984.

[3] V.I. Utkin. *Sliding Modes in Control Optimization.* Springer-Verlag, Berlin, Germany, 1992.

[4] V.I. Utkin, J. Guldner, and J. shi. *Sliding Mode Control in Electromechanical Systems.* Taylor and Francis, London (UK)and Philadephia (USA), 1999.

[5] W. Pierruquetti and J.P. Barbot. *Sliding Mode Control in Engineering.* Marcel Dekker, Inc, New York Basel, 2002.

[6] C. Edwards, E. Fossas, and L. Fridman. *Advances in Variable Structure Control and Sliding Mode Control.* Springer-Verlag, Berlin Heidelberg New York, 2006.

[7] A. Levant. Sliding order and sliding accuracy in sliding mode control. *International Journal of Control*, 1993.

[8] G. Bartolini, A. Ferrara, A. Pisano, and E. Usani. On the convergence properties of a 2-sliding control algorithm for nonlinear uncertain systems. *International Journal of Control*, 2001.

[9] Y.B. Shtessel, I. A. Shkolnikov, and Mark D.J. Brown. An asymptotic second order sliding smooth sliding mode control. *Asian Journal of Control*, 5(4):498–504, 2003.

[10] Y. Pan, K. Furuta, and S. Hatakeyama. Invariant sliding sector for variable structure control. In *Decision and Control, 1999. Proceedings of the 38th IEEE Conference on*, volume 5, pages 5152 –5157 vol.5, 1999.

[11] K. Furuta and Y. Pan. Variable structure control with sliding sector. *Automatica*, 36(2):211 – 228, 2000.

[12] Y. Pan, K.D. Kumar, G. Liu, and K. Furuta. Design of variable structure control system with nonlinear time-varying sliding sector. *IEEE Transactions on, Automatic Control*, 2009.

[13] A. Levant. Robust exact differentiation via sliding mode technique. *Automatica*, 1998.

[14] H. Sira-Ramirez. Dynamical sliding mode control strategies in the regulation of nonlinear chemical process. *International Journal of Control*, 1992.

[15] X. Lu and S. K. Spurgeon. Output feedback stabilization of siso nonlinear systems via dynamical sliding modes. *International Journal of Control*, 1998.

[16] X. Lu and S. K. Spurgeon. Output feedback stabilization of mimo nonlinear systems via dynamical sliding modes. *International Journal of Robust and Nonlinear Control*, 1999.

[17] V. Utkin and Jingxin Shi. Integral sliding mode in systems operating under uncertainty conditions. In *Decision and Control, 1996., Proceedings of the 35th IEEE*, volume 4, pages 4591 –4596 vol.4, dec 1996.

[18] Q. Khan, A.I. Bhatti, Q. Ahmed, and M. Iqbal. Dynamic integral sliding mode control of siso uncertain nonlinear systems. *International Journal of Innovative Computing, Information and Control*, 8(7), 2011.

[19] Q. Khan, A.I. Bhatti, and Q. Ahmed. Dynamic integral sliding mode control of nonlinear siso systems with states dependent matched and mismatched uncertainties. *IFAC*, 44(1):1105 –1120, aug 2011.

[20] Q. Khan, A.I. Bhatti, and A. Ferrara. Integral manifold based design of a dynamic sliding mode controller for siso nonlinear uncertain systems. *Under the review of Int. Journal of Control*, 2012.

[21] Q. Khan, A.I. Bhatti, S. Iqbal, and M. Iqbal. Dynamic integral sliding mode control of mimo uncertain nonlinear systems. *Int. Journal of Control Automation and Systems*, 44(1):1105 –1120, feb 2011.

[22] Q. Khan, A.I. Bhatti, and A. Ferrara. Integral manifold based design of a dynamic sliding mode controller for mimo nonlinear uncertain systems. *Under the review of IEE, Control theory and Applications*, 2012.

[23] S.V. Emelyanov. *Variable Structure Control Systems.* Nauka(in Russian), Moscow, 1967.

[24] Y. Itkis. *Control Systems of Variable Structure.* Wiley, New York, 1976.

[25] V.I. Utkin. Variable structure systems with sliding modes. *IEEE Transactions on, Automatic Control*, 1977.

[26] M. Fliess. Generalized controller canonical form for linear and nonlinear dynamics. *IEEE Transactions on, Automatic Control*, 35(9):994 –1001, sep 1990.

[27] H. Sira-Ramirez. Dynamical sliding mode control strategies in the regulation of nonlinear chemical process. *International Journal of Control*, 1993.

[28] A. Isidori. *Nonlinear Control Systems.* 3rd Edition, Springer-Verlag, 1995.

[29] J.J. Slotine and W. Li. *Applied Nonlinear Control.* Prentice-Hall, London:, 1991.

[30] A. Atassi and H. Khalil. Separation results for stabilization of nonlinear systems using different high-gain observer designs. *Systems and Control Letters*, 39:183–191, 2000.

[31] X. Yu and J.-X. Xu. Nonlinear derivative estimator. *Electronics Letters*, 32(16):1445 –1447, aug 1996.

[32] A. Levant. Higher order sliding modes, differentiation and output feedback control. *International Journal of Control*, 2003.

[33] K.D. Young, V.I. Utkin, and U. Ozguner. A control engineer's guide to sliding mode control. *IEEE Transactions on, Control Systems Technology*, 7(3):328 –342, may 1999.

[34] A. G. Bondarev, N. Y Kostylyeva, and V.I Utkin. Sliding modes in systems with asymptotic state observers. *Autom. Remote Control*, 1985.

[35] H. G. Kwatny and K. K. D Young. The variable structure servomechanism. *Systems and Control Letters*, 1(3):184 – 191, 1981.

[36] G. Bartolini, A. Ferrara, and E. Usani. Chattering avoidance by second-order sliding mode control. *IEEE Transactions on, Automatic Control*, 1998.

[37] G. Bartolini, A. Ferrara, A. Levant, and E. Usai. *Variable Structure Systems, Sliding Mode and Nonlinear Control.* in Lecture Notes in Control and Information Sciences, K. D. Young and U. Ozguner, Springer-Verlag, Berlin, Germany:, 1999.

[38] G. Bartolini, Punta E. Pisano, A., and E. Usai. Asurvey of applications of second order sliding mode control to mechanical systems. *International Journal of Control*, 2003.

[39] T. Floquet, J-P. Barbot, and W Perruquetti. Higher-order sliding mode stabilization for a class of nonholonomic perturbed systems. *Automatica*, 39(6):1077 – 1083, 2003.

[40] M. K. Khan and S. K. Spurgeon. Robust mimo water level control in interconnected twin-tanks system using second order sliding mode control. *Control Engineering Practice*, 14, 2006.

[41] S. Iqbal, C. Edwards, and A.I. Bhatti. A smooth second order sliding mode control for relative degree two systems. *IECON'10, Glendale, AZ, USA*, 2010.

[42] S. Iqbal, C. Edwards, and A.I. Bhatti. Robust feedback linearization using higher order sliding mode observers. *IEEE Control Conference on Decision and Control and European Control Conference, Orlando, Florida, USA*, 2011.

[43] Y. Shtessel and I. Shkolnikov. Aeronautical and space vehicle control in dynamic sliding manifolds. *International Journal of Control*, 2003.

[44] I. A. Shkolnikov and Y. B Shtessel. Tracking in a class of nonminimum-phase systems with nonlinear internal dynamics via sliding mode control using method of system center. *Automatica*, 38(5):837 – 842, 2002.

[45] A.S.I. Shtessel, Y.and Zinober and I. Shkolnikov. Sliding mode control of boost and buck-boost power converters using the dynamic sliding manifold. *International Journal of Robust and Nonlinear Control*, 2003.

[46] C. Edwards and S.K. Spurgeon. *Sliding Mode Control Theory and Applications*. Taylor and Francis, London, 1998.

[47] F.J. Bejarano, A. Poznyak, and L. Fridman. Mini-max integral sliding-mode control for multimodel linear uncertain systems. *IEEE Transactions on, Automatic Control*, 49(1).

[48] F.J. Bejarano, A. Poznyak, and L. Fridman. Min-max output integral sliding mode control for multiplant linear uncertain systems. *American Control Conference, New York City, USA, July*, 2007.

[49] Z. Chen. Integral sliding mode control of a bipedal leap. *International Journal of Robotics and Automation*, 22(3):252–259, 2007.

[50] J-L Chang. Dynamic output feedback integral sliding mode control design for uncertain systems. *International Journal of Robust and Nonlinear Control*, 22(8):841–857, 2012.

[51] A. Levant. Universal siso sliding mode controller with finite convergence. *IEEE Transactions on, Automatic Control*, 49(9):1447 –1451, sep 2001.

[52] J.C. Scarratt, A. Zinober, R. E. Mills, M. Rios-Bolivar, A. Ferrara, and L. Giacomini. Dynamical adaptive first and second-order sliding backstepping control of nontriangular uncertain systems. *Journal of Dynamic Systems, Measurement and Control*, 2000.

[53] D. Swaroop, J.K. Hedrick, P.P. Yip, and J.C. Gerdes. Dynamic surface control for a class of nonlinear systems. *IEEE Transactions on, Automatic Control*, 45(10):1893 – 1899, oct. 2000.

[54] A. Ferrara and L. Giacomini. On modular backstepping design with second order sliding modes. *Automatica*, 37(1):129–135, 2001.

[55] A. Estrada and L. Fridman. Quasi-continuous hosm control for systems with unmatched perturbations. *Automatica*, 46:1916–1919, 2010.

[56] A. Levant and L. Alelishvili. Integral high-order sliding modes. *IEEE Transaction on Automatic Control*, 52(7):1278–1282, 2007.

[57] G. Bartolini, A. Ferrara, and E. Usai. Output tracking control of uncertain nonlinear second order systems. *Automatica*, 33(12):2203–2212, 1997.

[58] Fridman L. Boiko, I. and M.I. Castellanos. Analysis of second-order sliding-mode algorithms in the frequency domain. *IEEE Transactions on, Automatic Control*, 49(6):946 – 950, sep 2004.

[59] A. Levant. Quasi-continuous high-order sliding mode controllers. *IEEE Transactions on, Automatic Control*, 2005.

[60] F. Dinuzzo and A. Ferrara. Higher order sliding mode controllers with optimal reaching. *IEEE Transactions on Automatic Control*, 54(9):2126–2136, 2009.

[61] H-H Choi. Variable structure output feedback control design for a class of uncertain dynamic systems. *Automatica*, 38(2):335 – 341, 2002.

[62] P. Park, D. J. Choi, and S. G Kong. Output feedback variable structure control for linear systems with uncertainties and disturbances. *Automatica*, 43(1):72 – 79, 2007.

[63] W. Xiang, J.and Wei and H. Su. An ilmi approach to robust output feedback sliding mode control. *International Journal of Control*, 2006.

[64] J.M.A.-D. Silva, C. Edwards, and S.K. Spurgeon. Sliding-mode output-feedback control based on lmis for plants with mismatched uncertainties. *IEEE Transactions on, Industrial Electronics*, 56(9):3675 –3683, sept. 2009.

[65] W-J. Cao and J-X Xu. Nonlinear integral-type sliding surface for both matched and unmatched uncertain systems. *IEEE Transactions on, Automatic Control*, 49(8):1355 – 1360, aug. 2004.

[66] F. Castanos and L. Fridman. Analysis and design of integral sliding manifolds for systems with unmatched perturbations. *IEEE Transactions on, Automatic Control*, 51(5):853 – 858, may 2006.

[67] M. Rubagotti, A. Estrada, F. Castanos, A. Ferrara, , and L. Fridman. Integral sliding mode control for nonlinear systems with matched and unmatched perturbations. *IEEE Transactions on Automatic Control*, 56(11):2699–2704, 2011.

[68] J-L. Chang. Dynamic output integral sliding-mode control with disturbance attenuation. *IEEE Transactions on, Automatic Control*, 54(11):2653 –2658, nov. 2009.

[69] M. Iqbal, A.I. Bhatti, S. Iqbal, and Q. Khan. Robust parameter estimation of nonlinear systems using sliding mode differentiator observer. *IEEE Transaction on Industrial Electronics*, 58(2):680–689, 2011.

[70] K-K. Shyu, Y-W. Tsai, and C-K Lai. A dynamic output feedback controllers for mismatched uncertain variable structure systems. *Automatica*, 37(5):775 – 779, 2001.

[71] J. Liu and F Sun. A novel dynamic terminal sliding mode control of uncertain nonlinear systems. *Journal of Control Theory and Applications*, 5:189–193, 2007. 10.1007/s11768-005-5275-5.