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Stabilization of Nonholonomic Systems

by

Waseem Abbasi

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Stabilization of Nonholonomic Systems

By

Waseem Abbasi

(PE123011)

Dr. Adrian Traian Plesca

Gheorghfe Asachi Technical University of Lasi, Romania

Dr. Jawad Masood

Automotive Technology Centre of Galicia (CTAG), Spain

Dr. Fazal ur Rehman

(Thesis Supervisor)

Dr. Noor Muhammad Khan

(Head, Department of Electrical Engineering)

Dr. Imtiaz Ahmed Taj

(Dean, Faculty of Engineering)

DEPARTMENT OF ELECTRICAL ENGINEERING

CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY

ISLAMABAD

2018

Dedicated to my beloved father Chan Khan Abbasi, Mother, Wife and Daughter

CAPITAL UNIVERSITY OF SCIENCE & TECHNOLOGY
ISLAMABAD
Expressway, Kahuta Road, Zone-V, Islamabad
Phone:+92-51-111-555-666 Fax: +92-51-4486705
Email: info@cust.edu.pk Website: https://www.cust.edu.pk

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Student Name: Mr. Waseem Abbasi (PE123011) ________________

The Examining Committee unanimously agrees to award PhD degree in the mentioned field.

Examination Committee :

(a)	External Examiner 1:	Dr. Nisar Ahmed Professor GIK Institute, Topi	________
(b)	External Examiner 2:	Dr. Adeel Mehmood Assistant Professor COMSATS University, Islamabad	________
(c)	Internal Examiner :	Dr. Aamir Iqbal Bhatti Professor CUST, Islamabad	________

Supervisor Name :	Dr. Fazal ur Rehman Professor CUST, Islamabad	________
Name of HoD :	Dr. Noor Muhammad Khan Professor CUST, Islamabad	________
Name of Dean :	Dr. Imtiaz Ahmad Taj Professor CUST, Islamabad	________

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(Mr. Waseem Abbasi)

Registration No : PE123011

Dated: 02 October, 2018

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Registration No. PE123011

Dated: 02 October, 2018

List of Publications

It is certified that following publication(s) have been made out of the research work that has been carried out for this thesis:-

Journal Publications

1. **Waseem Abbasi** and Fazal ur Rehman, "Adaptive Integral Sliding Mode Stabilization of Nonholonomic Drift-Free Systems", Mathematical Problems in Engineering 2016 (2016).
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Abstract

The stabilization problem of nonholonomic systems, for many reasons, has been an active research topic for the last three decades. A key motivation for this research stems from the fact that nonholonomic systems pose considerable challenges to control system designers. Nonholonomic systems are not stabilizable by smooth time-invariant state-feedback control laws, and hence, the use of discontinuous controllers, time-varying controllers, and hybrid controllers is needed. Systems such as wheeled mobile robots, underwater vehicles, and underactuated satellites are common real-world applications of nonholonomic systems, and their stabilization is of significant interest from a control point of view. Nonholonomic systems are, therefore, a principal motivation to develop methodologies that allow the construction of feedback control laws for the stabilization of such systems.

In this dissertation, the stabilization of nonholonomic systems is addressed using three different methods. The first part of this thesis deals with the stabilization of nonholonomic systems with drift and the proposed algorithm is applied to a rigid body and an extended nonholonomic double integrator system. In this technique, an adaptive backstepping based control algorithm is proposed for stabilization. This is achieved by transforming the original system into a new system which can be asymptotically stabilized. Once the new system is stabilized, the stability of the original system is established. Lyapunov theory is used to establish the stability of the closed-loop system. The effectiveness of the proposed control algorithm is tested, and the results are compared to existing methods.

The second part of this dissertation proposes control algorithm for the stabilization of drift-free nonholonomic systems. First, the system is transformed, by using input transformation, into a particular structure containing a nominal part and some unknown terms that are computed adaptively. The transformed system is then stabilized using adaptive integral sliding mode control. The stabilizing controller for the transformed system is constructed that consists of the nominal control plus a compensator control. The Lyapunov stability theory is used to derive the compensator control and the adaptive laws. The proposed control algorithm is

applied to three different nonholonomic drift-free systems: the unicycle model, the front-wheel car model, and the mobile robot with trailer model. Numerical results show the effectiveness of the proposed control algorithm.

In the last part of this dissertation, a new solution to stabilization problem of nonholonomic systems that are transformable into chained form is investigated. The smooth super twisting sliding mode control technique is used to stabilize nonholonomic systems. Firstly, the nonholonomic system is transformed into a chained form system that is further decomposed into two subsystems. Secondly, the second subsystem is stabilized to the origin using the smooth super twisting sliding mode control. Finally, the first subsystem is steered to zero using the signum function. The proposed method is applied to three nonholonomic systems, which are transformable into chained form; the two-wheel car model, the model of front-wheel car, and the firetruck model. Numerical computer simulations show the effectiveness of the proposed method when applied to chained form nonholonomic systems.

This research work is mainly focused on the design of feedback control laws for the stabilization of nonholonomic systems with different structures. For this purpose, the methodologies adopted are based upon adaptive backstepping, adaptive integral sliding mode control, and smooth super twisting sling mode control technique. The control laws are formulated using Lyapunov stability analysis. In all cases, the control laws design for the transformed models is derived first, which is then used to achieve the overall control design of the kinematic model of particular nonholonomic systems. Numerical simulation results confirm the effectiveness of these approaches.

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Abbreviations

SMC	Sliding Mode Control
SEV	Space Exploration Vechiles
WMRs	Wheeled Mobile Robots
SOSM	Second Order Sliding Mode
HOSM	Higher Order Sliding Mode
VSC	Variable Structure Control
SM	Sliding Mode
SS	Sliding Surface
RP	Reaching Phase
ISMC	Integral Sliding Mode Control
AISMC	Adaptive Integral Sliding Mode Control
UMS	Underactuated Mechanical System
ENDI	Extended Nonholonomic Double Integrator
STA	Super-Twisting Algorithm
SSTA	Smooth Super-Twisting Algorithm

Symbols

x, y	state vectors
n	order of the system
s	sliding surface
u, v, w	control inputs
m	order of control input
$f(x)$	system function
$d(u)$	system input function
c	constant value
M	gain of the sliding surface
t	time
u_{eq}	equivalent control input
λ	design constant for control gain
ρ	control design parameter
$[f, g]$	Lie bracket
θ	steering angle
α	virtual control
V	Lyapunov function

Chapter 1

Introduction

In the early decades of the $20th$ century, developments in the design of control systems were mainly focused in the area of linear control. All the systems to be controlled were assumed to be linear or approximated by linear time-invariant differential equations. In most of the cases, where the linear approximation techniques accurately describe the system, linear control methods are successful. However, in many cases, the systems are too nonlinear and cannot be correctly described by linear differential equations. In such cases, nonlinear techniques are required for exact system representation. For example, in case of nonholonomic systems, nonlinear control techniques are more appropriate as the constraints on nonholonomic systems are inherently nonlinear.

Nonholonomic systems are defined as "systems that satisfy certain non-integrable constraints". The constraints may arise as the result of system's physical dynamics and can be expressed in term of the generalized velocities. These constraints on nonholonomic systems are not integrable, i.e., the constraints cannot be written as the time derivative of generalized coordinates. This type of constraints on generalized velocities is considered as first-order nonholonomic constraints and such systems are known as first order nonholonomic systems.

In the last few decades, the applications area of the nonholonomic systems has grown immensely due to practical utilization of these systems in different fields.

As far as the applications are concerned, it is hardly possible to avoid contact with nonholonomic systems. There are several examples of such systems in our daily life, like going away to work by driving a car, pushing a stroller, riding a bike, parking of a tractor with many trailers and so on.

One of the crucial features of nonholonomic systems is that these systems are not amenable to a method of linear control theory, and hence, fundamentally require nonlinear techniques for stabilization. Another essential characteristic of nonholonomic systems is that the inputs are always less than the number of its degrees of freedom, i.e., these systems belong to the family of underactuated mechanical systems (UMS). These features make the stabilization and control of nonholonomic systems even more challenging and difficult.

Nonholonomic systems are controllable, but at some instant, they cannot move in some specific directions. Although, nonholonomic systems are allowed to move in any direction at certain time, but due to mechanical constraint, the so-called nonholonomic constraints or kinematic constraints, they are unable to move to a particular position or state at a particular instant, e.g. wheeled vehicles, that move only in perpendicular direction to the axle connecting the wheels. In [1], the author also highlights this problem that nonholonomic systems are not stabilizable to the origin by continuous static state feedback laws. To solve this stabilization problem, different control schemes have been adopted in the literature, such as discontinuous control [2], time-varying control [3] and hybrid control [4]. Using these techniques, the stabilization and tracking control of these systems can be achieved.

Many nonlinear techniques, such as feedback linearization, gain scheduling, adaptive control, adaptive backstepping and sliding mode control have been developed for the stabilization and tracking control problems of nonholonomic systems. In feedback linearization, the control laws are designed in such a way that they cancel the nonlinear terms, resulting closed-loop system in a linear form. The linear controller then stabilizes the resulting closed-loop system. This scheme may apply to many nonlinear systems, but it cannot be applied to nonholonomic systems.

1.1 Motivation

Before the invention of nonholonomic systems and their practical use, they were just limited to the stuff of some scientific and academic terminologies. However, nowadays, a vast variety of these systems are being deployed for their useful use in daily life. Many applications of nonholonomic systems like advanced robotic structures, wheeled mobile robots, and multi-finger robotic hands have gained attention due to their role in different practical systems. In these systems, the nonholonomic behavior can be controlled entirely with fewer numbers of actuators than the degrees of freedom.

Nonholonomic systems pose new control problems that require fundamental nonlinear approaches. The linear approximation of these systems around equilibrium points may not be controllable, and the feedback linearization technique cannot transform the system into a linear control problem. Consequently, the linear control techniques cannot be adopted to solve the feedback stabilization or the tracking control problem. However, under certain conditions, the feedback stabilization and tracking control problem can be solved by time-varying control or the discontinuous feedback control strategies.

Many mechanical systems are subject to constraints on their generalized velocities. In the field of classical mechanics, these constraints are also defined as linear constraints of the type $\phi(q)\dot{q} = 0$ which are not integrable, where the generalized coordinates are denoted by q. The constraints are said to be non-integrable if it cannot be written as the function of the generalized coordinates, i.e., $\psi(q) = 0$, and thus cannot be solved by integration. Contrary to classical mechanics, a more general characterization of nonholonomic constraints is chosen in this thesis.

In addition to classical formulations of nonholonomic systems, nonholonomic constraints can arise in other ways. If the motion of a mechanical system exhibits specific symmetry properties, for example, if the angular momentum of the mechanical system is not integrable, this may be interpreted as a nonholonomic constraint. It should be noted that, in classical mechanics, conserved quantities are not regarded as constraints on a system. In control community, however, it has been commonly accepted to view these conserved quantities as constraints that are imposed on the system. Examples of such systems include multi-body systems and underactuated symmetric rigid spacecraft. Nonholonomic constraints also arise as a result of imposing design constraints on the allowable motions of the mechanical systems like kinematically redundant manipulators and underactuated manipulators. An introduction and overview of nonholonomic control systems can be found in [5].

Nonholonomic systems are an excellent platform for research and educational purpose. The kinematic constraints called "Nonholonomic constraints" imposed on these systems make their control a challenging problem. If we talk about selected examples of nonholonomic systems, a two-wheel car model, a front-wheel car model, a car with trailer model, and a firetruck model are good examples of nonholonomic systems having nonholonomic constraints. These mobile robots are highly subject to external disturbance, e.g., slippery floor, dusty air, and design of control laws for such systems in the presence of external disturbances or uncertainties is a crucial task.

1.2 Research Objectives

In the light of motivation and problems arising in the stabilization of nonholonomic systems [1]; an objective is to investigate novel and more effective feedback control design methodologies for the stabilization of nonholonomic systems. Moreover, due to the existence of parametric uncertainties, the controller design for stabilization of nonholonomic systems will become much more difficult due to the simultaneous existence of nonholonomic constraints and unknown system parameters. The methodologies must be general and applicable to whole class of nonholonomic systems instead to some specific system. Meanwhile, in the absence of smooth static time-invariant feedback laws, the primary objective is to develop such techniques that are based on:

1. Discontinuous feedback control laws

2. Time-varying feedback control laws

Finally, we will validate the methodologies for the following benchmark nonholonomic systems:

1. Rigid Body

2. Extended Nonholonomic Double Integrator

3. A Unicycle Model

4. A Front Wheel Car Model

5. A Car with Trailer Model

6. A Firetruck Model

1.3 Research Contributions

In this thesis, the stabilization problem of nonholonomic systems is considered. Namely, the class of nonholonomic systems that can be transformed, by an appropriate coordinate and feedback transformation, into a particular form. The particular form considerably simplifies the kinematic equations of the system and is, therefore, more suitable for control design than the original kinematic model. To date, studies on the control of nonholonomic systems have primarily been focused on motion planning and trajectory tracking, while less interest has been focused on the stabilization problem. Although the kinematic models of nonholonomic systems are quite well understood, the stabilization problem of these systems remains a challenging task.

In this thesis the stabilization problem of the nonholonomic systems is solved with three different techniques:

i. Adaptive Backstepping Control [6].

ii. Adaptive Integral Sliding Mode Control (AISMC) [7].

iii. Smooth Super-Twisting Sliding Mode Control (SSTSMC) [8].

The stabilization of nonholonomic systems by using adaptive backstepping, AISMC, and SSTSMC in the presence of constraints is the subject of this research. Finding general control laws based directly on a kinematic model will make understanding, analysis, and implementation of these laws straightforward. Furthermore, closed-form analytic expressions for design parameters are used throughout the control schemes, which is another consideration in this thesis. It also addresses the transformation from the kinematic model of nonholonomic systems into a particular form. The stability of the nonholonomic systems is analyzed with the help of Lyapunov functions. Furthermore, this thesis includes a complete methodology for controller design of nonholonomic systems by providing:

- Transformation of the kinematic model of nonholonomic systems into particular form by using input and state transformation [7, 8]
- Design of the control laws based on an adaptive backstepping technique to stabilize the nonholonomic system [6]
- AISMC laws based on a Lyapunov function which stabilizes the nonholonomic systems [7].
- SSTSMC laws based on a Lyapunov function which stabilizes the nonholonomic systems in a chained form [8].

1.4 Outline of the Thesis

This thesis consists of seven chapters, and Figure 1.1 highlights the chapter wise structure of this dissertation. The focus of this research work is to propose novel solution to the stabilization problem of nonholonomic systems. In order to organize a self-contained text on stabilization of such systems, various feedback techniques are being reviewed to highlight the significance of stabilization problem in the presence of constraints. Finally, the framework is presented in a detailed fashion and numerical results are being examined.

The descriptions of the individual chapters are as follows:

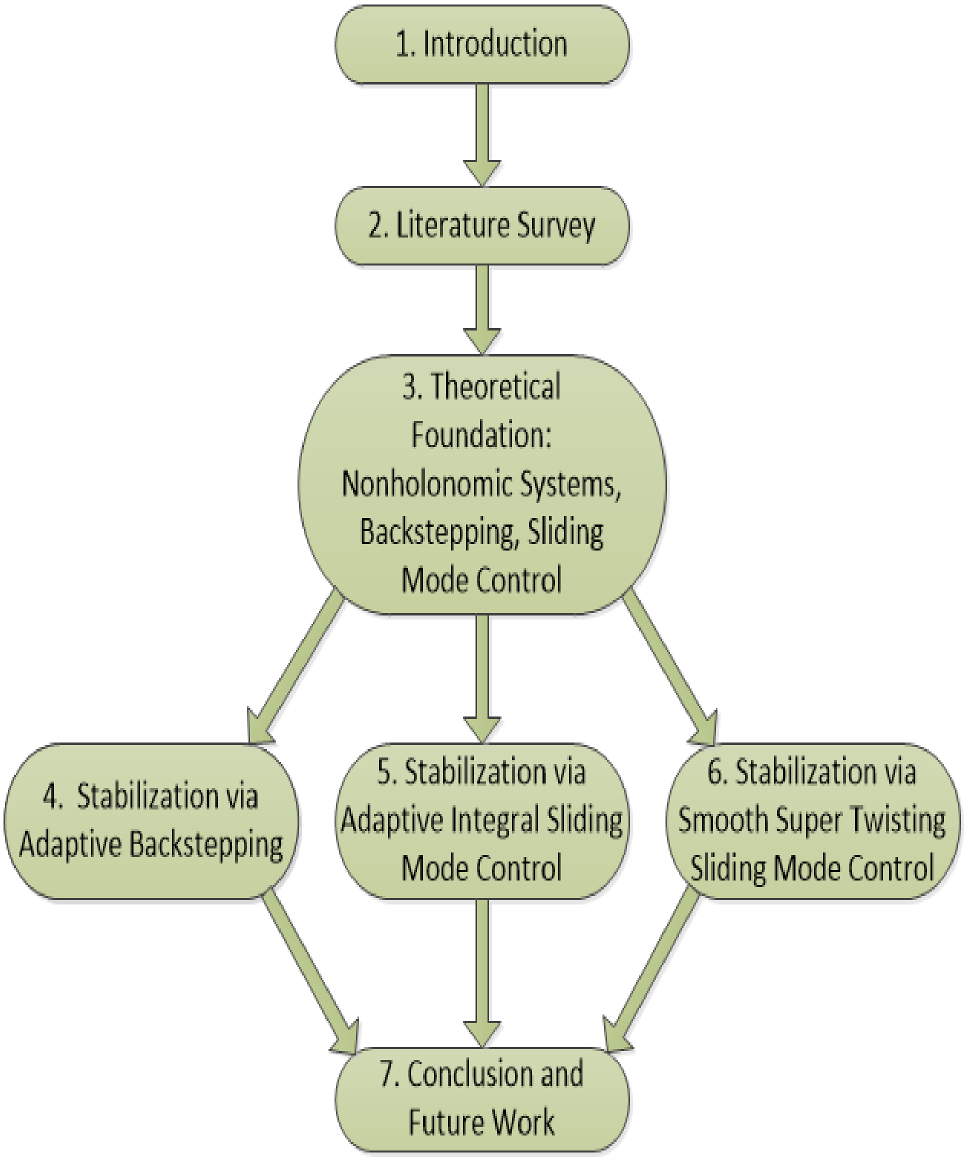

FIGURE 1.1: Thesis flow chart

- **Chapter 2, Literature Survey:** This chapter takes into account the literature review regarding stabilization of nonholonomic systems. We summarize the literature with the findings of limitations and contributions of various stabilization techniques. The research gaps help us to formulate the problem statement and to find the general laws which help in finding the solution for the stabilization problem of nonholonomic systems.

- **Chapter 3, Theoretical Foundation:** In this chapter some preliminaries are provided to understand the stabilization problem of nonholonomic systems. This chapter provides foundation for Chapter 4, Chapter 5 and Chapter 6.
- **Chapter 4, Stabilization of Nonholonomic Systems: Adaptive Backstepping Technique:** This chapter presents adaptive backstepping based stabilization of the nonholonomic system with drift to create an interest and insight in the stabilization of nonholonomic systems. The algorithm presented is applied to a rigid body and an extended nonholonomic double integrator which is considered as a benchmark nonholonomic system. The chapter also highlights the result along with a comparison with the existing results in the literature.
- **Chapter 5, Stabilization of Nonholonomic Systems: Adaptive Integral Sliding Mode Control Technique:** This chapter presents the proposed control algorithm based on adaptive integral sliding mode control. The algorithm is applied to nonholonomic wheeled mobile robots like, e.g., a unicycle car model, a front wheel car model, and a model of car with a trailer.
- **Chapter 6, Stabilization of Nonholonomic Systems: Smooth Super Twisting Sliding Mode Control Technique:** This chapter presents the proposed control algorithm based on smooth super twisting sliding mode control. The algorithm is applied to nonholonomic wheeled mobile robots model in chained form e.g., a unicycle car, a front wheel car, and a firetruck model.
- **Chapter 7, Conclusion and Future Work:** This chapter summarizes the overall thesis and draws conclusions along with the recommendations and claims. The significance of the proposed research is emphasized. Future directions have also been proposed for further research.

1.5 Summary

In this chapter, first, the aim is to present a sketch and motivations behind this research. Secondly, the research objectives are defined on the basis of overview and motivation from existing literature. Thirdly, the contribution of this research work is formulated on the basis of research objective. At the end, an outline of the research work is also discussed.

Chapter 2

Literature Survey

This chapter aims to review different stabilization approaches to highlight the stabilization problem and to identify the gaps, weaknesses, and issues in the stabilization of nonholonomic systems. Another aspect of this chapter is to review the different stabilization methodologies that researchers apply to investigate and to help the industry in a choice of schemes. Furthermore, this literature review is a meta-analysis of stabilization problem of nonholonomic systems. It enables to integrate the findings to enhance the understanding of the stabilization problem and research gaps in order to formulate the problem statement.

2.1 Stabilization of Nonholonomic Systems

Nonholonomic systems belong to a particular class of nonlinear systems and are frequently encountered in the real world. Applications of such systems are car-like and underwater vehicles, mobile robots, satellites and so on. However [1] showed that stabilization of nonholonomic systems could not be achieved by smooth static state feedback laws although, these systems are controllable. As a consequence, well-developed time-invariant continuous nonlinear control methodologies cannot be directly applied.

To overcome the difficulty imposed by Brockett's condition [1], different control approaches have been presented in the literature. The construction of feedback control laws for stabilization of nonholonomic systems offers some exciting challenges. As such systems are inherently nonlinear; linear control techniques cannot be used. Moreover, such systems are not transformable into linear systems (even locally) in any meaningful form [2]. As a result, nonlinear control techniques or non-classical definitions of the solution to differential equations are needed.

Nonholonomic systems are usually controllable; but at some instant, they cannot move in specific directions, e.g., wheeled vehicles, which moves only in a perpendicular direction to the axle connecting the wheels. Due to the mechanical constraints on wheeled vehicles, the problem of a parallel parking of the car becomes an exciting challenge. Consider the car performing a parking maneuver at some distance from its current position in the same orientation (see Figure 2.1). This type of behavior is a particular case of nonholonomic systems.

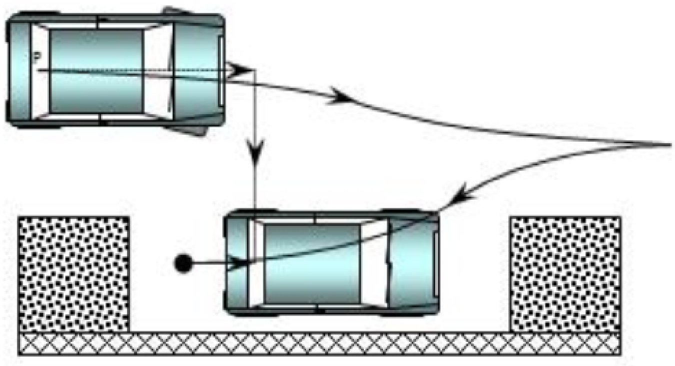

FIGURE 2.1: A car in parking maneuver: cannot move sideways

Nonholonomic systems are a typical class of systems with nonlinear nature. Due to these nonlinearities, nonholonomic systems are difficult to stabilize. In order to cater this problem, many strategies have been proposed for the stabilization problem of nonholonomic control systems. The work of Kolmanovsky and McClamroch [5] serves as a tutorial presentation for the developments in the control of nonholonomic systems.

A general introduction of nonholonomic systems and constraints on the nonholonomic systems was, firstly, discussed by Murray, Li, and Sastry [9]. The book by Siciliano, Sciavicco, Villani and Oriolo [10] also provide an excellent literature, where the concepts are introduced in a coherent and didactic way. This book also contains techniques for modeling, planning, and control of the nonholonomic systems having kinematic constraints. It also states the basic concepts from differential geometry applied to holonomic and nonholonomic systems, lead to finding the controllability of these systems.

For the stabilization of nonholonomic systems, tools of differential geometry for a mathematical solution were first used more than 150 years ago. Systematic development of the theory was initiated by mathematicians and physicists, based on a series of classic articles by Chaplygin, Carathodory, and Hertz. Recently, the formation of control laws for such systems was discussed extensively. The differential geometric approach in the context of nonholonomic systems was introduced by Brockett, Isidori, Sussmann, and others in the 1970's. Since then, the advancement of theoretical tools to the design of feedback laws for the wide class of nonholonomic systems have been done.

The kinematic model-based control of nonholonomic systems addresses the problem of motion planning, trajectory tracking, and point stabilization of such systems. The control models for such systems are usually nonlinear. One approach to deal with nonholonomic systems is to use time-varying controllers [11]. In [11], it has been shown that time-varying controllers for nonholonomic systems provided algebraic (not exponential) convergence. For global asymptotic stabilization of nonholonomic systems using time-varying law is obtained in [12].

Another way to stabilize such systems is to use discontinuous controllers that show exponential convergence [13]. In [14], a discontinuous piecewise control scheme was proposed to achieve exponential convergence. Later on, discontinuous control law was proposed to obtain the exponential convergence along the desired trajectory [15]. The control law was implemented on a nonholonomic chained form system and achieved global stabilization. However, the system was not point stabilizable with the proposed control law. Moreover, the extraction of discontinuous, adaptive and the robust control law is a demanding and challenging job in the presence of unknown parameters, external disturbances, and uncertainties.

The stabilization of nonholonomic systems with different control techniques was presented in [6, 16–19]. A feedback control law is proposed for *nth* order trailer model is presented in [20]. In [21], a hybrid control approach is implemented on the nonholonomic car-like robot with uncertainty. The author in [22] proposed a technique based on sliding mode control for a broad group of nonholonomic systems suffering from drift uncertainties. Furthermore, the design of globally asymptotic controllers for the stabilization problem of these systems is a challenging task, and multiple control design strategies have been used for this purpose.

The stabilization problem for nonholonomic systems can be solved by two broad class of techniques, namely: open loop control and closed-loop control. The open-loop control steers the nonholonomic systems in a given finite interval from initial to final state. Various differential geometric tools were also being utilized to make control laws for open-loop control of nonholonomic systems in [23]. In [24], a sinusoid technique has been proposed to stabilize nonholonomic systems. Open loop techniques for nonholonomic systems of low dimensions have been proposed in [25]. A generalization of the sinusoids method by Lie bracket is also discussed in [26]. Furthermore, in [27], a piecewise constant input and polynomial input are further elaborated for the stabilization of nonholonomic systems.

Despite open loop, closed-loop control techniques, such as feedback linearization, have gained much attention in the early 1980's for the stabilization of nonholonomic systems. These systems have uncontrollable linearization for smooth, continuous feedback control laws. Differential geometric methods were used for approximate feedback linearization for a broad class of nonholonomic systems in [28, 29]. Later, with the development of control Lyapunov functions and backstepping techniques [30], a more generalized solution in the presence of matched and unmatched uncertainties was given for stabilization of such systems.

The model of many nonholonomic systems, like wheeled mobile robots, underwater vehicles, and satellites are first transformed into some meaningful form with the help of input or state transformation. During the past few decades, researchers and scientists worked rigorously to find such transformations. The discontinuous or time-varying control was then applied to the transformed systems to stabilize. Discontinuous control showed exponential stability [31–33], while the time-varying control, led to the asymptotic stability of such systems [34, 35]. More recent studies show the exponential stability of the transformed nonholonomic systems with the help of time-varying homogeneous control [36].

These transformations usually result in the discontinuous control laws, which need a switching control to keep the system states away from the singularity hyperplane. The discontinuous control proposed in [13, 37, 38] describe ρ-process and state scaling, which results in the exponential stabilization of nonholonomic systems. In [39, 40], improvements were made to cater the singularity problem due to singular initial conditions. Although these existing state transformations provide a good solution, there is still an open question to develop a global nonsingular state transformation that maps the nonholonomic systems into a controllable form which yield global exponential stabilization.

The theory of nonholonomic open loop and closed-loop control states that there exist no continuous, differentiable time-invariant control laws, which can asymptotically stabilize the nonholonomic systems. Therefore, many researchers proposed time-varying control laws [41, 42], discontinuous control laws [43, 44], and hybrid control laws [45, 46] for the stabilization of nonholonomic systems. For the stabilization of these systems, there are two broader control strategies namely:

i. Open loop control

ii. Closed-loop control

2.1.1 Open Loop Control

The open-loop control also referred under the name of motion planning, stabilizes the nonholonomic systems in a given finite interval from initial to final state. Some of the open-loop control techniques like model predictive control for the stabilization of nonholonomic systems have been discussed in [47]. In [48], an excellent information about open loop control for nonholonomic robots was discussed. The open-loop control algorithm can be further classified into following three groups according to which mathematical models are derived.

i. Strategies derived from employing differential-geometric and algebraic techniques.

ii. Strategies based on the special control parametrization.

iii. Strategies employing methods of optimal control.

The open-loop control techniques also include inputs like sinusoidal, piecewise constant, and a polynomial. The underlying logic behind sinusoidal inputs is to steer all states one by one by using a sinusoid. The polynomial input approach is more favorable than the piecewise constant input approach, because of its smoothness. The open-loop control has one major drawback that model error and disturbances occur in a system model cannot be eliminated. To cater to this problem a closed loop or feedback control approach is used.

2.1.2 Closed Loop or Feedback Control

A closed-loop control system is also known as a feedback control system. The term "feedback" simply means that some portion of the output is returned "back" to the input. As far as stabilization problem of the nonholonomic systems is concerned, a feedback law must be designed around an equilibrium point. In [49], a nonholonomic system model was transformed into a two-dimensional model to obtain a discontinuous feedback law. This discontinuous feedback controller enforces the nonholonomic system to asymptotic approach to a particular circle which contains the origin. In [50–52], non-smooth state feedback laws were used to overcome the problem of stabilization of nonholonomic systems.

In [53, 54], sliding mode approach was proposed, which used discontinuous feedback control to stabilize nonholonomic systems. The sliding mode control approach uses discontinuous laws, which force the trajectory to slide along the manifold, but the main disadvantage is that it produces chattering. In the absence of smooth static feedback, the closed-loop synthesis method concentrates on the following strategies.

i. Discontinuous state feedback

ii. Time-varying state feedback

iii. Hybrid feedback control

2.1.2.1 Discontinuous State Feedback Control

One of the methods to stabilize nonholonomic systems is to use discontinuous control, which is quite simpler than its counterpart, time-varying control. However, one major drawback of this technique is the singularity manifold, which is due to the state transformation. The main logic behind the discontinuous control is to switch the control, as system states leave the singular manifold. In [2], a ρ-process is proposed to represent such control law. Discontinuous state feedback control can be further categorized into two types.

i. Piecewise continuous feedback

ii. Sliding mode control

(i) Piecewise Continuous Feedback Control

The piecewise continuous feedback controller for a broad class of nonholonomic systems is presented in [55]. In this article, an exponential convergence with the help of piecewise continuous feedback law has been presented for some specific examples of nonholonomic systems. For the attitude stabilization of an underactuated rigid spacecraft with a piecewise continuous feedback law has been proposed in [56]. A different way for constructing a piecewise continuous controller has been proposed in [57, 58]. A saturated finite-time stabilization for systems in feed-forward form is proposed in [59].

(ii) Sliding Mode Control

A sliding mode control (SMC) is well-known nonlinear feedback control technique. One of the main advantages of SMC approach is, the system becomes insensitive to disturbance and parameter variations. The SMC forces the trajectory to slide along the particular surface to reach the equilibrium point. Compared to other control laws, SMC is robust and is easy to implement. The SMC approach for a large class of nonholonomic mechanical systems was proposed in [60, 61]. The SMC approach also used for a particular class of higher-order nonholonomic systems in [62, 63]. A comprehensive and detailed literature on SMC technique is studied in the next chapter.

2.1.2.2 Time-Varying State Feedback Control

Another method to stabilize nonholonomic systems is to use time-varying controllers which provide asymptotic convergence. A time-varying control law for stabilization problem of such systems was presented in [64]. The stabilization of the kinematic model of nonholonomic systems at the equilibrium point by time-varying control was presented in [65]. Some explicit feedback laws for global asymptotic stabilization of nonholonomic systems was presented in [66]. A smooth time-varying feedback control, which explicitly depends upon time variables were presented in [67]. In [67, 68], the smooth feedback laws are constructed by averaging and saturation function. A smooth feedback law with asymptotic convergence rate has been presented in [69].

In [70], linear controllers were constructed, which stabilize a nonholonomic system on the nominal trajectory. For a construction of "nominal trajectory", the asymptotic stabilizing scheme has been presented in [71]. A smooth time periodic feedback laws were generated by using Pomet's method to construct a suitable

Lyapunov function was presented in [69]. Furthermore, [72] demonstrated that the nonholonomic system could not be steered to a neighborhood in desired time by smooth time-periodic feedback laws. Thus, there is a need for faster convergence rate laws, which necessarily be non-smooth. More detailed information on convergence rate and the smoothness of feedback law can be found in [73].

For the stabilization problem of dynamic nonholonomic systems, a time-periodic feedback law is proposed from kinematic controllers using integrator backstepping techniques in [6]. The stabilization of a subclass of nonholonomic systems via a time-varying feedback control has been discussed in [64]. In [74, 75], knife-edge, underactuated rigid spacecraft, and free-floating multi-body spacecraft have been stabilized by the time-varying controller.

2.1.2.3 Hybrid Feedback Control

The hybrid feedback control is the combination of discrete time and continuous time features. As presented in [76], the hybrid controller is based on switching between low-level continuous-time control laws. An excellent introduction to the hybrid controllers for stabilization of the nonholonomic system like a mobile robot was given in [77]. The hybrid controllers for attitude stabilization of an underactuated rigid spacecraft have been proposed in [78]. Exponential stabilization of these systems in a chained form by using a hybrid controller was first presented in [79]. A more general hybrid controller, which is applicable to a large class of nonholonomic systems was proposed in [80, 81].

2.2 Findings of the Literature Survey

The literature was reviewed in two directions (open loop control and closed loop control) to cover majority aspects of the stabilization problem of nonholonomic systems. It also emphasizes to establish the significance of stabilization of such systems in the research field and in robotic industry, where new contributions could be made. The bulk of this literature review was on the evaluation of the different methodologies used in the stabilization of nonholonomic systems. It also helps us to identify the appropriate methods for investigating the research questions raised in the previous chapter. The researchers are striving in all the areas and phases to explore novel techniques to stabilize such systems. In the field of stabilization or tracking control of such systems, it can be concluded that there is no single suitable technique available for the smooth static feedback stabilization of nonholonomic systems. The researchers are relying on the discontinuous and time-varying methods, which globally stabilize such systems. Furthermore, hybrid techniques are also being applied to address the stabilization problem. Following are the list of difficulties or research gap in existing literature, which help to clarify the scope or the objective of this dissertation.

- Existing methods, such as feedback stabilization and motion planning used diffeomorphic state transformation, which converts the system into either chained form or power form [2, 11, 82, 83]. These transformations are usually defined locally.
- Many time-varying feedback approaches relay on the construction of Lyapunov function [3, 34, 84], which is not easy to find.
- In asymptotic stabilization, time-varying controller does not provide exponential convergence [55].
- One disadvantage of robust techniques like sliding mode control, is that it produces unwanted oscillation known as chattering [85].
- Consequently, approaches for the design of robust control laws for nonholonomic systems are less known. Open loop approaches are fewer likely to produce solutions, which are not robust with respect to modeling uncertainties and sensor error as compared to feedback approaches.

2.3 Summary

In this chapter, the stabilization problem of nonholonomic systems was presented to pinpoint the gaps and limitations in existing literature. Regardless of particular forms and techniques, a variety of control techniques are available which have their pros and cons as mentioned above. Furthermore, when we reviewed the literature on the stabilization of nonholonomic systems, almost all the methods which have been applied to the stabilization problem has certain assumptions. Consequently, based on the review of nonholonomic systems, a complete framework is in dire need to cope with the constraints such that these systems can be asymptotically stabilized at origin or their equilibrium points.

Chapter 3

Theoretical Foundation

In this chapter, theoretical foundation and the techniques to solve the stabilization problem of nonholonomic systems are discussed in detail. The terminologies and research techniques that are helpful to develop the proposed methodologies are explained with the help of simulation examples.

3.1 The Constraints Model of Nonholonomic Systems

Nonholonomic systems belong to a special class of nonlinear systems having non-integrable constraints on their velocities. Such constraints may arise in a number of ways; few examples of nonholonomic systems having constraints are given below:

Wheeled Mobile Robots:

The drive mechanisms of wheeled mobile robot results in constraints on the instantaneous velocities. For example, a wheeled robot with two forward drive wheels and two back wheels is often required to move without slipping sideways.

Hoping Robots:

In case of hoping robots, the constraints arise because of angular momentum conservation, as there is no external force applied on such a system.

Space Robotics:

For unmanned aerial vehicle (UAV), the thrust is always aligned with the longitudinal direction of the body, so there is no slippage.

3.1.1 Linear Velocity Constraints and their Integrability

Most of the velocity constraints mentioned above have the form of linear constraints expressed by the following system of equation:

$$a_i(q)\dot{q} = 0, \quad q \in \Re^n, \quad i = 1, ..., k \tag{3.1}$$

where the vector $q \in R^n$ describes the configuration of the system to be controlled. The following example illustrates how linear velocity constraints can be written in the form of Eq. (3.1).

Front Wheel Car Model:

Consider the front wheel drive car model as shown in Fig. 3.1. The front tires can spin about the vertical axes while the rear tires are aligned with the car. Let $q = \begin{pmatrix} \psi \; x \; y \; \theta \end{pmatrix}^T$ be the configuration vector, where, ψ is the directing angle w.r.t the car body, (x, y) is the position of the rear wheels, θ is the angle of the car body w.r.t the horizontal and l is the distance between the front and the rear wheels.

The constraints on this system are that the wheels can only move in a perpendicular direction to the axle connecting the wheels. The velocity is always restricted to wheels aligned with the heading angle, as no slippage is allowed. The constraints for the front and rear wheels can be expressed by making the velocities of the wheels (sideways) to zero. The velocity of the front wheels perpendicular to the direction is $\sin(\theta + \psi)\,\dot{x} - \cos(\theta + \psi)\,\dot{y} - l\cos\psi\dot{\theta}\cdot = 0$, and velocity of the back wheels perpendicular to their direction is $\sin\theta\,\dot{x} - \cos\theta\,\dot{y} = 0$. These constraints

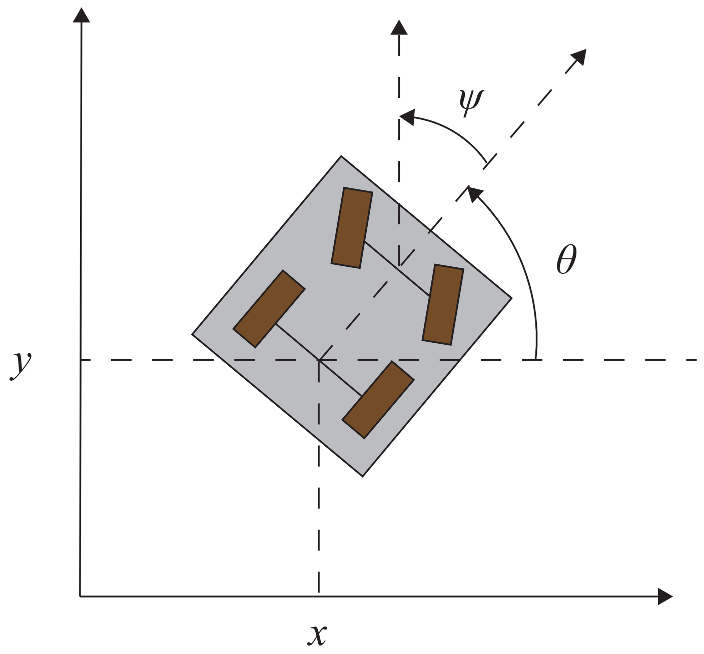

FIGURE 3.1: A front wheel car model

can be thus written as:

$$\sin\theta\,\dot{x} - \cos\theta\,\dot{y} = 0 \quad \text{or} \quad a_1^T(q)\dot{q} = 0 \tag{3.2}$$

$$\sin(\theta+\psi)\,\dot{x} - \cos(\theta+\psi)\,\dot{y} - l\cos\psi\dot{\theta} = 0 \quad \text{or} \quad a_2^T(q)\dot{q} = 0 \tag{3.3}$$

where

$$a_1^T(q) = [\,0 \quad \sin\theta \quad -\cos\theta \quad 0],$$

$$a_2^T(q) = [0 \quad \sin(\theta+\psi) \quad -\cos(\theta+\psi) \quad -l\cos\psi\,]$$

or in matrix form:

$$A^T(q) = \begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \sin(\theta+\psi) & -\cos(\theta+\psi) & -l\cos\psi & 0 \end{bmatrix} \tag{3.4}$$

The constraints are said to be integrable if for each q there exist scalar functions $h_j : N(q) \rightarrow R, \quad j = 1, 2, ..., k,$ (defined on some neighbor hood $N(q)$ of q), such that Eq. (3.1) can be written as:

$$\frac{d}{dt}h_j(q) = \nabla h_j(q)\dot{q} = 0, \quad j = 1, 2, ..., k, \quad \text{for} \quad \mathrm{q} \in \mathrm{N(q)} \tag{3.5}$$

where ∇ denotes the gradient of h_j. Integrating Eq. (3.5) yields:

$$h_i(q) = 0 \quad j = 1, 2, ..., k, \quad q \in N(q) \tag{3.6}$$

It follows that integrable constraints can be substituted by algebraic constraints which do not involve velocities. The constraints are said to be non-integrable if they cannot be written as algebraic constraints involving only configuration variables q. Integrable constraints are known as holonomic constraints and non-integrable constraints are called nonholonomic constraints.

3.2 The Kinematic Model of Nonholonomic Systems

The nonholonomic systems have restricted motion in a particular direction due to the constraints. Because of these constraints, we have to discuss the directions in which movement is permitted, instead of discussing the directions in which movement is not allowed. Consider the problem of parallel parking of the car, making a path $q(t) \in \Re^n$ between given points $q(0) = q_0$ to $q(t_f) = q_f$, in presence of constraints $A^T(q)\,\dot{q} = 0$.

The kinematic constraint $A^T(q)\,\dot{q} = 0$ essentially implies that the motion of configuration are in the null space of constraints $A^T(q)$ i.e., $\dot{q} \in \mathrm{null}\{\mathrm{A^T(q)}\}$, then there exists $n - k$ smooth linearly independent vector fields $g_i(q)$, $i = 1, 2, ..., n - k$ such that:

$$\mathrm{null}\{\mathrm{A^T(q)}\} = \Delta(\mathrm{q}) = \mathrm{span}\{\mathrm{g_1(q)}\,, \mathrm{g_2(q)}, \, ..., \, \mathrm{g_{n-k}(q)}\} \tag{3.7}$$

It is now clear that the nonholonomic constraints $a_i^T(q)\,\dot{q} = 0$ are equivalent to the statement that $\dot{q} \in \Delta(q) = \mathrm{span}\{\mathrm{g_1(q)}\,, \mathrm{g_2(q)},\, ...,\, \mathrm{g_{n-k}(q)}\}$, which requires that:

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2 + \ldots + g_{n-k}(q)u_{n-k} \tag{3.8}$$

with coefficients $u_1, u_2, ..., u_{n-k}$ generally depend on time t, as q and $\dot{q}$ vary with time. The above equation represents a control system, in which q is the state vector and $u_1, u_2, ..., u_{n-k}$ are the control inputs. Assuming that the velocity $\dot{q}$ can be actuated directly, the path planning problem becomes as, finding the control $u(t) = \begin{bmatrix} u_1(t) \; u_2(t) \cdots u_{n-k}(t) \end{bmatrix}^T \in \Re^{n-k}$ which steers q_0 to q_f. Therefore, the kinematic model of nonholonomic systems can be written as:

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2 + \ldots + g_{n-k}(q)u_m = \sum_{i=1}^{m} g_i(q)u_i \tag{3.9}$$

The constraints on the front wheel car model are:

$$A^T(q)\,\dot{q} = 0$$

where

$$A^T(q) \;=\; \begin{bmatrix} 0 & \sin\theta & -\cos\theta & 0 \\ 0 & \sin(\theta+\psi) & -\cos(\theta+\psi) & -l\cos\psi \end{bmatrix}$$

Then the null space of $A^T(q)$ is given by:

$$\mathrm{null}\,\{A^T(q)\,\} = \mathrm{span}\{\mathrm{g_1(q)}, \mathrm{g_2(q)}\}$$

where, $g_1(q) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$ & $g_2(q) = \begin{bmatrix} 0 & \cos\theta & \sin\theta & \frac{1}{l}\tan\psi \end{bmatrix}^T$.

Therefore the kinematic model of front wheel car can be written as:

$$\dot{q} = \begin{bmatrix} \dot{\psi} \\ \dot{x} \\ \dot{\theta} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \\ \frac{1}{l}\tan\psi \end{bmatrix} u_2$$

3.3 Important Features of Nonholonomic Systems

In [1], the author proved that nonholonomic systems cannot be stabilized by time-invariant feedback control laws. The exact linearization of nonholonomic systems is impossible. Similarly, point-wise linearization also fails, because the system is not controllable at the origin.

The important features of the nonholonomic systems are explained as:

1. The number of inputs is always less than the number of states variable, and hence these systems are similar to underactuated mechanical systems.
2. The linearization of Eq. (3.9) at any point is uncontrollable.
3. The Eq. (3.9) has no equilibrium point i.e giving $u_1 = ... = u_m = 0$ results in $\dot{x} = 0$.
4. The controllability of nonholonomic systems can be guaranteed by Chow's theorem [86]:

$$\mathrm{span}\{\mathrm{g_i(q), g_j(q), \ldots, [g_i, g_j](q), \ldots}\} = \mathbb{R}^{\mathrm{n}}$$

Due to these features, nonholonomic systems are the special class of nonlinear system and the problem of stabilization of nonholonomic systems becomes a challenging task. In this chapter, an introduction to different approaches have been explained to solve this problem e.g., backstepping, adaptive backstepping, sliding mode control, super twisting sliding mode control, and adaptive integral sliding mode control.

3.4 Backstepping and Adaptive Backstepping

In control system theory, backstepping became a useful tool for the design of control laws that ensure the stability of a special class of nonlinear systems. A design of this technique is made up of straightforward fundamental steps. This technique, start with the first state variable with the maximum number of integrations and "step back" towards the controller. The whole procedure is completed when we reach the final control. That is why this process is known as "backstepping". The "backstepping" procedure gives us a method for the stability of the system in a "strict-feedback form". Backstepping itself cannot solve the uncertainty problem. The control difficulties caused by these uncertainties can be removed using adaptive method along with the backstepping which is known as adaptive backstepping. The adaptive backstepping procedure is illustrated by the following example.

Example:

Consider the following system:

$$\dot{x}_1 = x_2 \tag{3.10a}$$

$$\dot{x}_2 = x_3 + \theta \tag{3.10b}$$

$$\dot{x}_3 = u \tag{3.10c}$$

Where θ is unknown. Let $\hat{\theta}$ be the estimate of θ and $\tilde{\theta} = \theta - \hat{\theta}$ be the parameter error. By Eq. (3.10a): $\dot{x}_1 = x_2$. Consider x_2 as a virtual control and λ_1 as stabilizing function, then the error variable is: $z_1 = x_2 - \lambda_1$. Eq. (3.10a) becomes:

$$\dot{x}_1 = z_1 + \lambda_1 \tag{3.11}$$

Consider the Lyapunov function: $V_0 = \frac{1}{2}x_1^2$ for computing the stabilizing function. Then

$$\dot{V}_0 = x_1\dot{x}_1 = x_1(z_1 + \lambda_1) \tag{3.12}$$

Choose $\lambda_1 = -x_1$. Then

$$\dot{V}_0 = -x_1^2 + x_1 z_1 \tag{3.13}$$

The term $x_1 z_1$ will be canceled in the next step. Eq. (3.11) becomes:

$$\dot{x}_1 = z_1 - x_1 \tag{3.14}$$

Now $z_1 = x_2 + x_1 \implies \dot{z}_1 = \dot{x}_2 + \dot{x}_1 = x_3 + \hat{\theta} + \tilde{\theta} + z_1 - x_1$. Consider x_3 state as virtual control and λ_2 as stabilizing function, the error variable becomes $z_2 = x_3 - \lambda_2$. Then

$$\dot{z}_1 = z_2 + \lambda_2 + \hat{\theta} + \tilde{\theta} + z_1 - x_1 \tag{3.15}$$

Consider the Lyapunov function: $V_1 = V_0 + \frac{1}{2}z_1^2$ for computing the stabilizing function λ_2. Then

$$\begin{aligned}\dot{V}_1 = \dot{V}_0 + z_1\dot{z}_1 &= -x_1^2 + x_1 z_1 + z_1(z_2 + \lambda_2 + \hat{\theta} + \tilde{\theta} + z_1 - x_1) \\ &= -x_1^2 + z_1\{z_2 + \lambda_2 + \hat{\theta} + \tilde{\theta} + z_1\}\end{aligned} \tag{3.16}$$

Choose $\lambda_2 = -2z_1 - \hat{\theta}$, Then

$$\dot{V}_1 = -x_1^2 - {z_1}^2 + z_1 z_2 + z_1\tilde{\theta} \tag{3.17}$$

The term $z_1 z_2$ will be canceled in the next step. Eq. (3.15) becomes:

$$\dot{z}_1 = z_2 - z_1 - x_1 + \tilde{\theta} \tag{3.18}$$

and, $z_2 = x_3 - \lambda_2 = x_3 + 2z_1 + \hat{\theta} \implies \dot{z}_2 = \dot{x}_3 + 2\dot{z}_1 + \dot{\hat{\theta}}$

$$\dot{z}_2 = u + 2(z_2 - z_1 - x_1 + \tilde{\theta}) + \dot{\hat{\theta}} \tag{3.19}$$

Consider the Lyapunov function $V_2 = V_1 + \frac{1}{2}z_2^2 + \frac{1}{2}\tilde{\theta}^2$. Then

$$\begin{aligned}
\dot{V}_2 &= \dot{V}_1 + z_2\dot{z}_2 + \tilde{\theta}\dot{\tilde{\theta}} = \dot{V}_1 + z_2\{u + 2(z_2 - z_1 - x_1 + \tilde{\theta}) + \dot{\hat{\theta}}\} + \tilde{\theta}\dot{\tilde{\theta}} \\
&= -x_1^2 + z_1z_2 - {z_1}^2 + z_1\tilde{\theta} + z_2\{u + 2z_2 - 2z_1 - 2x_1 + 2\tilde{\theta} + \dot{\hat{\theta}}\} + \tilde{\theta}\dot{\tilde{\theta}} \\
&= -x_1^2 - z_1^2 + z_2\{2z_2 - z_1 - 2x_1 + \dot{\hat{\theta}} + u\} + \tilde{\theta}\{\dot{\tilde{\theta}} + z_1 + 2z_2\} \qquad (3.20)
\end{aligned}$$

Choose $\dot{\tilde{\theta}} = -z_1 - 2z_2 - \tilde{\theta}$, and $u = -3z_2 + z_1 + 2x_1 - \dot{\hat{\theta}}$. Then $\dot{V}_2$ becomes:

$$\dot{V}_2 = -x_1^2 - z_1^2 - z_2^2 - \tilde{\theta}^2 \qquad (3.21)$$

Eq. (3.19) becomes, $\dot{z}_2 = -z_2 - z_1 + 2\tilde{\theta}$. The closed loop systems becomes:

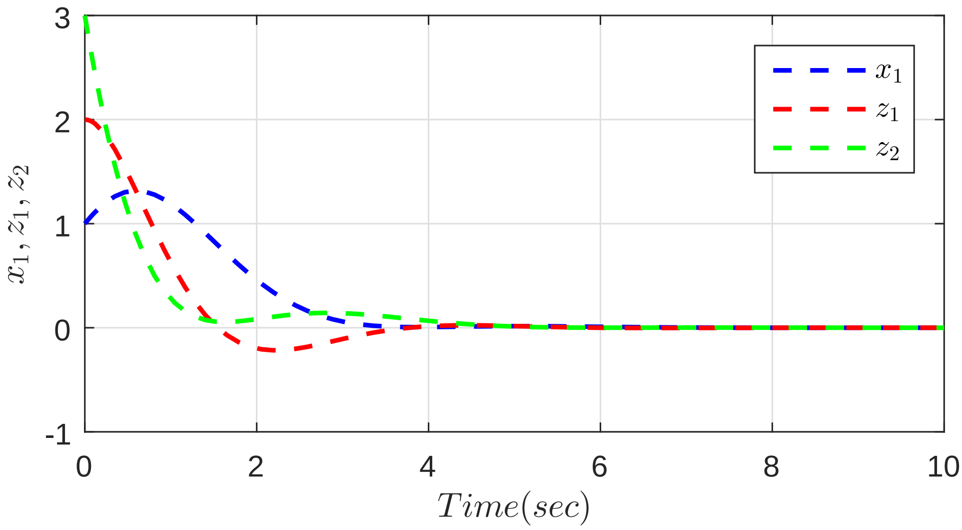

FIGURE 3.2: Closed loop response of system: $(x_1, z_1, z_2) = (1, 2, 3)$

$$\dot{x}_1 = z_1 - x_1 \qquad (3.22a)$$

$$\dot{z}_1 = z_2 - z_1 - x_1 + \tilde{\theta} \qquad (3.22b)$$

$$\dot{z}_2 = z_2 - z_1 + 2\tilde{\theta} \qquad (3.22c)$$

$$\dot{\tilde{\theta}} = -z_1 - 2z_2 - \tilde{\theta} \qquad (3.22d)$$

$$\dot{\hat{\theta}} = -\dot{\tilde{\theta}} \qquad (3.22e)$$

Figure 3.2 shows the response of the closed loop systems.

3.5 Sliding Mode Control

Among the well-known nonlinear control techniques, Sliding Mode Control (SMC) has gained considerable attention in the last few decades. It provides remarkable features like accuracy, robustness and easy tuning capability and implementation. The control design process in SMC can be divided into two parts.

i. Design of sliding surface in such a way that the desired response of the system is achieved.

ii. Selection of suitable control law, such that in the presence of external disturbances and model uncertainties, it drags the sliding variable to zero.

Design of the sliding surface is established by keeping in view the design specifications of sliding motion. After designing the sliding surface, a suitable law is needed, to attract the system states towards the switching surface [61]. Closed loop response in SMC consists of the following phases:

i. A reaching phase in which plant trajectories are driven to the sliding manifold.

ii. A sliding phase in which plant trajectories slides to the origin.

The main advantage of an SMC is its robustness against external disturbances and parameter variations. To stabilize the system, the gain is kept higher than the norm of external disturbances and parametric variations. The switching nature results in the undesirable phenomena known as chattering in SMC. Different Higher-Order Sliding Mode (HOSM) control techniques are also adopted for the elimination of the chattering. However, in the case where system dynamics are not known, an observer-based SMC law is formulated to stabilize the system.

The first main advantage of SMC is its dynamic behavior, which is due to the suitable selection of the sliding function. Secondly, for some particular uncertainties, closed-loop system response becomes insensitive. The principle of Sliding Mode

(SM) can be extended to the model which has disturbances, parameter uncertainties, and bounded nonlinearities. The control strategies such as SMC of uncertain systems were briefly discussed in [87]. The main principles of SMC are outlined in the important references regarding control of mechanical systems [88–90] and a table is also given in this regard. This comparison shows that sliding mode control strategies present an efficient way to control mechanical systems.

Table 3.1: Problems in the control of mechanical systems, and solutions proposed by smc [91].

Problem in Control of Mechanical Systems	**Solutions proposed by SMC**
Mechanical systems are nonlinear and strongly coupled.	Sliding mode control offers decoupling and order reduction.
Mechanical systems with unknown parameters, e.g., friction coefficients, loads.	Sliding mode control shows robustness against parameter uncertainties and unknown disturbances.

3.5.1 Sliding Mode Control Design Procedure

Consider the following nonlinear system:

$$\dot{x} = f(x) + g(x)u \tag{3.23}$$

where $x \in \Re^n$ is the state vector, $f(x) \in \Re^n$ and $g(x) \in \Re^n$ are the vector fields, where $u \in \Re$ represents the control input. The sliding surface $s(x)$ is chosen as:

$$s(x) = c^T x \tag{3.24}$$

where $c \in R^n$. Then

$$\dot{s}(x) = c^T \dot{x} \tag{3.25}$$

or

$$\dot{s}(x) = c^T(f(x) + g(x)u)$$
$$\dot{s}(x) = c^T f(x) + c^T g(x)u \tag{3.26}$$

The reaching law for Eq. (3.26) is defined in such a way that it ensures the establishment of sliding mode.

$$\dot{s}(x) = -Msign(s(x)) \tag{3.27}$$

where M is selected high as compared to the magnitude of disturbances and uncertainties. From Eq. (3.26), the control law can be define as:

$$u = -[c^T g(x)]^{-1}[c^T f(x) + Msign(s(x))] \tag{3.28}$$

or we can write the control law as:

$$u = -\left[c^T g(x)\right]^{-1} c^T f(x) - [c^T g(x)]^{-1} Msign(s(x)) \tag{3.29}$$

where $[c^T g(x)] \neq 0$, the control law has two components, one is the equivalent control component, and other is the discontinuous control component. The disturbances and uncertainties always exist in a practical system, and in such situation, the discontinuous control guarantees robustness. Figure. 3.3 shows the reaching phase (RP), sliding mode (SM) and the sliding surface (SS).

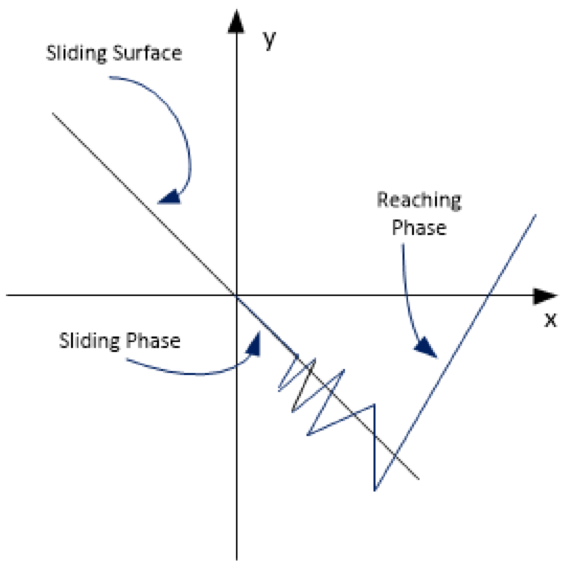

FIGURE 3.3: Phases of system motion in smc.

3.5.2 Chattering Phenomena

In sliding mode control, the control signal exhibits high-frequency oscillation called chattering. The chattering phenomena have negative effect on real-world applications and may lead to large unwanted oscillations that degrade the performance of the system.

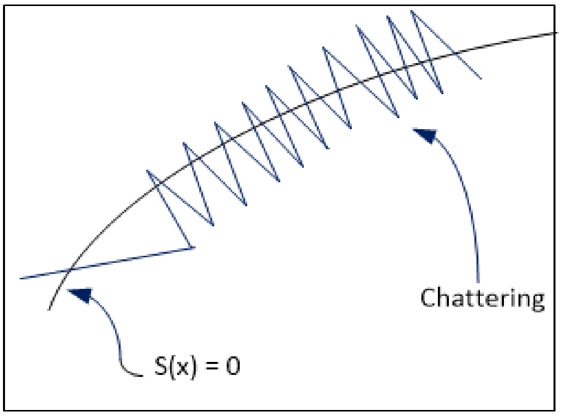

FIGURE 3.4: The chattering phenomena.

To avoid chattering phenomena, various solutions to this problem have been proposed [92, 93]. A new design scheme based on sliding variable was presented in [94]. In [95], a describing function approach was developed for chattering analysis on the system in the presence of the unmodeled dynamics. Another way to reduce chattering effect is by using second-order sliding mode (SOSM) and higher order sliding mode (HOSM) control techniques. Figure. 3.4 shows the chattering phenomena.

3.5.3 Integral Sliding Mode Control

In Integral Sliding Mode Control (ISMC), the reaching phase is eliminated and the robustness of the motion in the whole state space is guaranteed [96–98]. The ISMC consists of the nominal control input to stabilizes the nominal system and a discontinuous control input to rejects the uncertainties.

Consider the nonlinear system:

$$\dot{x} = f(x) + Bu + h(x,t) \tag{3.30}$$

where $x \in \Re^n$, $u \in \Re^m$ are states and input vectors, where $m < n$, $f(x)$ is the function having nonlinearities in the states, $B \in \Re^{n*m}$ is the input gain with span(B) = m. The matched uncertainty is represented by $h(x,t)$, such that $h(x,t) \in$ span(B), or in other words we can say $h(x,t) = Bu_h$. Also $h(x,t)$ is assumed to be bounded. The controller for integral sliding mode control can be expressed as:

$$u = u_0 + u_1 \tag{3.31}$$

where $u_0 \in \Re^m$ is the nominal controller and $u_1 \in \Re^m$ has two parts $u_1 = u_{1eq} + u_{1dis}$. u_{1eq} is designed in such a way that it rejects the disturbance $h(x,t)$ and $u_{1dis} = ksign(s)$. By putting (3.31) into (3.30) we get:

$$\dot{x} = f(x) + Bu_0 + Bu_1 + h(x,t) \tag{3.32}$$

The sliding surface is defined as:

$$s = s_0(x) + z \tag{3.33}$$

where $s \in \Re^m$ is the sliding surface, $s_0 \in \Re^m$ is the nominal sliding surface and $z \in \Re^m$ is the integral term chosen in such a way that $z(0) = -s_0(0)$. Differentiating (3.33) w.r.t time as:

$$\dot{s} = \dot{s}_0(x) + \dot{z}$$

$$\dot{s} = \frac{\partial s_0}{\partial x}\dot{x} + \dot{z}$$

$$\dot{s} = \frac{\partial s_0}{\partial x}[f(x) + Bu_0 + Bu_1 + h(x,t)] + \dot{z} \tag{3.34}$$

To achieve the goal of the integral sliding mode, the ideal and nominal part trajectory $x_0(t)$ must be equal to integral sliding mode trajectory $x(t)$, or in another way $x(t) = x_0(t)$. To fulfill the above-mentioned criteria, the equivalent control

$u_1(t)$ represented by $u_{1eq}(t)$ must fulfill the conditions as follows.

$$Bu_{1eq}(t) = -h(x,t) \tag{3.35}$$

The equivalent control can best describe the system state trajectories. The integral term z can be calculated using (3.35) into (3.34) as:

$$\begin{aligned} \dot{s} &= \frac{\partial s_0}{\partial x}[f(x) + Bu_0 + Bu_1 + h(x,t)] + \dot{z} \\ 0 &= \frac{\partial s_0}{\partial x}[f(x) + Bu_0] + \dot{z} \end{aligned} \tag{3.36}$$

Eq. (3.36) hold true if the following condition satisfied.

$$\dot{z} = -\frac{\partial s_0}{\partial x}[f(x) + Bu_0] \tag{3.37}$$

To ensure $s = 0$ the initial condition for $z(0)$ should be set accordingly, such that the sliding mode will exist right from the beginning. The system dynamics in Integral Sliding Mode (ISM) is:

$$\dot{x} = f(x) + Bu_0. \tag{3.38}$$

The system in Eq. (3.38) has no disturbance.

3.5.4 Higher Order Sliding Mode Control

The HOSM are popular for excellent performance and ease of implementation. The SOSM and HOSM control techniques reduce chattering effects [99–101]. Let r be sliding order of the system, and s be sliding variable, then higher order sliding mode ensures that $(r-1)th$ derivatives become zero of the sliding variable, i.e.,

$$s = \dot{s} = \ddot{s}..... = s^{(r-1)} = 0 \tag{3.39}$$

The SOSM algorithms, e.g., super twisting controller and the suboptimal algorithm were first time derived in [102, 103]. In standard or first order SMC, $\dot{s}$ is discontinuous, whereas, in SOSM $\ddot{s}$ is discontinuous and s is continuous. The chattering effect is minimized by finding a trade-off between disturbance rejection and the magnitude of the controller gain. The SOSM controller ensures convergence of $\dot{s} = 0$ within a finite time.

3.5.4.1 Super Twisting Algorithm

The super twisting algorithm [104] is a particular form of SOSM that gives finite time convergence. The twisting algorithm can be written as:

$$\begin{aligned} u &= -\lambda_1 |\sigma|^{1/2} \text{sign}(\sigma) + w \\ \dot{w} &= -\lambda_2 \text{sign}(\sigma) \end{aligned} \tag{3.40}$$

3.5.4.2 Smooth Super Twisting Algorithm

The smooth super twisting algorithm smoothens the chattering effect and does not require information regarding derivative of s. A smooth super twisting algorithm can be written as:

$$\begin{aligned} u &= -\lambda_1 |\sigma|^{\frac{\rho-1}{\rho}} \text{sign}(\sigma) + w \\ \dot{w} &= -\lambda_2 |\sigma|^{\frac{\rho-2}{\rho}} \text{sign}(\sigma) \end{aligned} \tag{3.41}$$

In real-world applications, sliding mode control often causes high switching frequencies which results in a wear and tear of actuators. The super twisting algorithm presents an efficient solution to this challenge. Another strategy for reducing chattering is to use asymptotic observers. However, this approach increases complexity as compared to conventional sliding mode. For an efficient chattering reduction, a twisting algorithm and super twisting algorithm are used.

Example 3.1. *Consider the second order non linear system with matched disturbance as shown below in Eq.* (3.42)*:*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \beta \sin(x_1) + u + d(x_1, x_2, t) \end{aligned} \tag{3.42}$$

TABLE 3.2: Result summary of third order system subject to chattering, robustness, and complexity [91]. The column symbols (+) represents low, moderate (++), and high (+++).

Approach	Chattering reduction	Robustness	Design complexity
Sliding mode control	+	+++	+
Observer-based control	+++	+++	+++
Hybrid control	++	+	++
Second order sliding mode control	++	+	+++
Twisting algorithm	+++	++	+++
Super twisting algorithm	+++	++	++

where $x_1 = x$ *and* $x_2 = \dot{x}$ *represent the system states,* u *is the control input and* $d(x_1, x_2, t)$ *represents the disturbance term with known upper bound* $|d(x_1, x_2, t)| \leq D$*. Whereas,* β *represents the physical parameter of the system.*

The objective is to define the control input u *in such a way that the overall states* x_1 *and* x_2 *will be asymptotically stabilized.*

A. Conventional Sliding Mode Control:

Suppose we want the nonlinear system (3.42) to behave like a linear system with the desired dynamics given by the following homogeneous linear time-invariant differential equation:

$$\dot{x}_1 + cx_1 = 0 \tag{3.43}$$

where $c > 0$ *is a design constant. In the first design step, we define the following sliding surface:*

$$\sigma = \sigma(x_1, x_2) = x_2 + cx_1 \tag{3.44}$$

Next, to achieve the desired dynamics in (3.44), in the presence of uncertainties/disturbance choose:

$$\sigma = x_2 + cx_1 = 0 \tag{3.45}$$

The derivative of the sliding variable σ *in* (3.44) *is represented as:*

$$\dot{\sigma} = cx_2 + \beta \sin(x_1) + u + d(x_1, x_2, t) \tag{3.46}$$

By defining the Lyapunov function for system (3.44)*:*

$$V = \frac{\sigma^2}{2} \tag{3.47}$$

Taking the derivative of V *we get:*

$$\dot{V} = \sigma\dot{\sigma} = \sigma(cx_2 + \beta \sin(x_1) + u + d(x_1, x_2, t)) \tag{3.48}$$

Let $u = -cx_2 - \beta \sin(x_1) - \Gamma sign(\sigma)$*, and substituting it in* (3.48) *we obtain:*

$$\dot{V} = \sigma(-\Gamma sign(\sigma) + d(x_1, x_2, t)) = -\sigma\Gamma sign(\sigma) + \sigma d(x_1, x_2, t) \leq -\sigma\Gamma sign(\sigma) + |\sigma| D \tag{3.49}$$

where $\Gamma > D$ *is a design constant and* $sign(x)$ *is defined as:*

$$sign(x) = \begin{cases} 1 & if \quad x > 0 \\ -1 & if \quad x < 0 \end{cases} \tag{3.50}$$

or Eq. (3.49) *is represented as:*

$$\dot{V} \leq -|\sigma|\Gamma + |\sigma|D = -|\sigma|(\Gamma - D) \tag{3.51}$$

which can be written as:

$$\dot{V} \leq -|\sigma|(\Gamma - D) = -\frac{\gamma}{\sqrt{2}}|\sigma| \tag{3.52}$$

Simulation Results:

Fig. 3.5 - Fig. 3.7 show simulation results with controller parameters $\gamma = 1$*,* $c = 1$*, and system parameter* $\beta = 1$ *for initial condition* $x_1(0) = 1$ *and* $x_2(0) = 2$*, disturbance term* $D(x_1, x_2, t) = cos(2\pi f t)$*,* $f = 0.5 cycles/sec$*.*

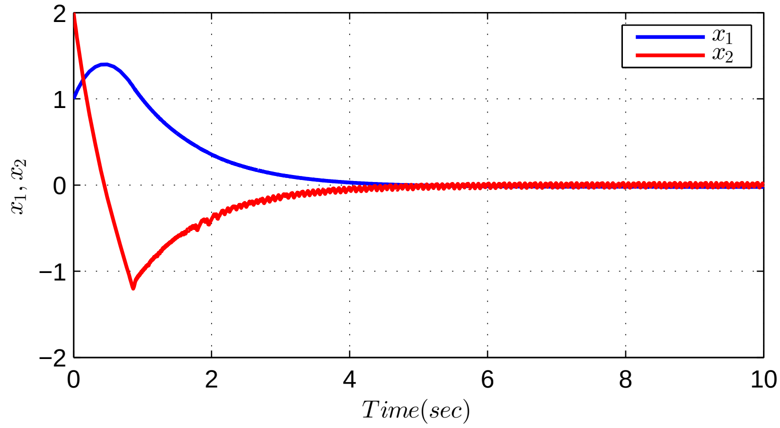

FIGURE 3.5: Closed loop response of system states for initial condition: $(x_1, x_2) = (1, 2)$

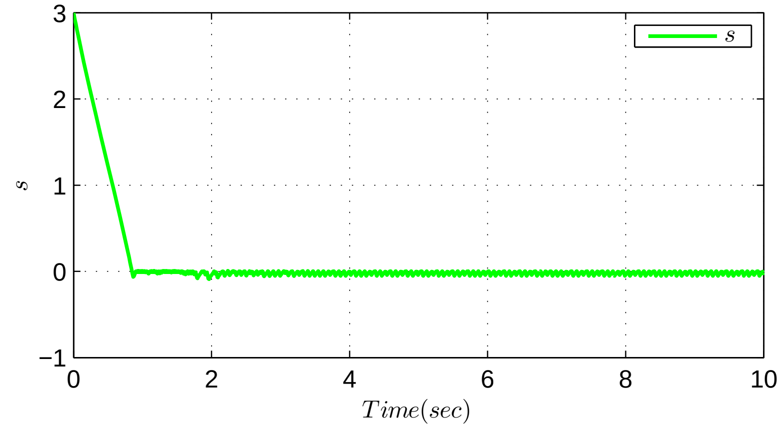

FIGURE 3.6: The sliding surface

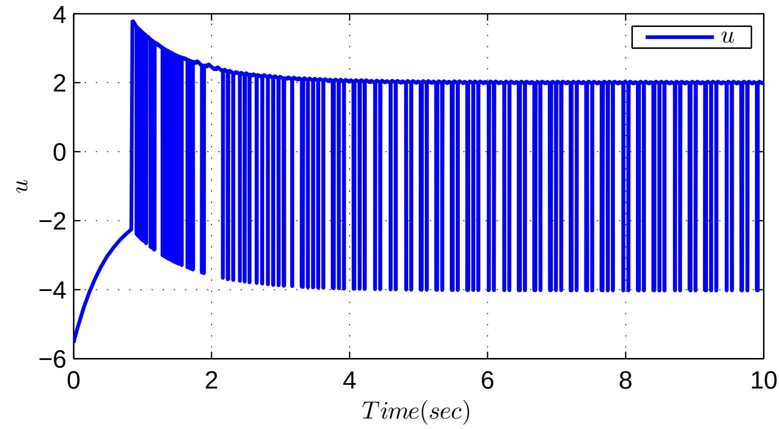

FIGURE 3.7: The control effort

As shown in the above simulation results, the designed conventional SMC law suffers from the practical issue of chattering.

B. Second Order Sliding Mode Control - The Super-Twisting Algorithm (STA):

To overcome the problem of Chattering, second order sliding mode control was introduced as an effective solution. We consider the following second order sliding based controller called the Super-Twisting Algorithm.for the example case.

$$\begin{aligned} u &= -\lambda_1 |\sigma|^{1/2} sign(\sigma) + w \\ \dot{w} &= -\lambda_2 sign(\sigma) \end{aligned} \tag{3.53}$$

where σ *is the same sliding variable defined in* (3.44) *and* λ_1 *and* λ_2 *are design constants for the controller gains.*

Simulation Results:

Fig. 3.8 - Fig. 3.10 show simulation results with controller parameters $\lambda_1 = 5$, $\lambda_2 = 8$, $c = 1$, *and system parameter* $\beta = 1$ *for initial condition* $x_1(0) = 1$ *and* $x_2(0) = 2$.

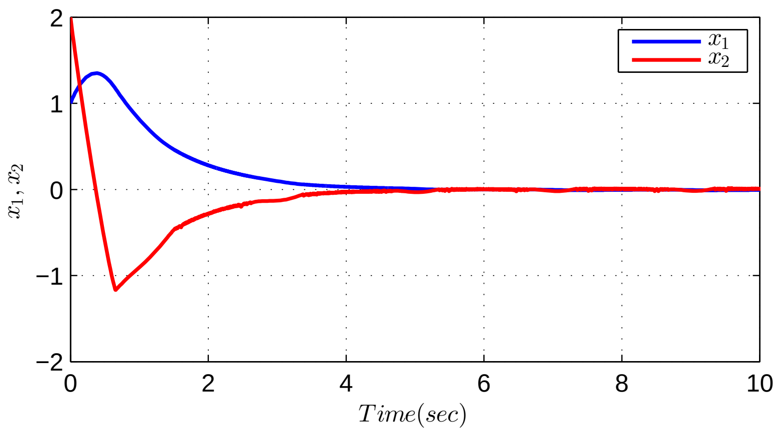

FIGURE 3.8: Closed loop response of system states for initial condition: $(x_1, x_2) = (1, 2)$

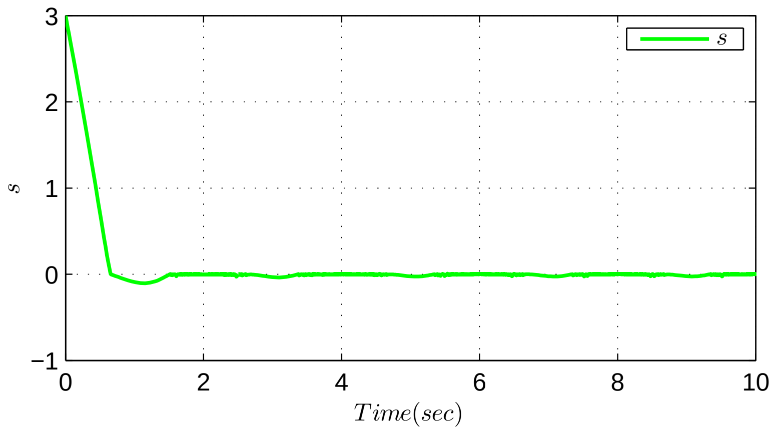

FIGURE 3.9: The sliding surface

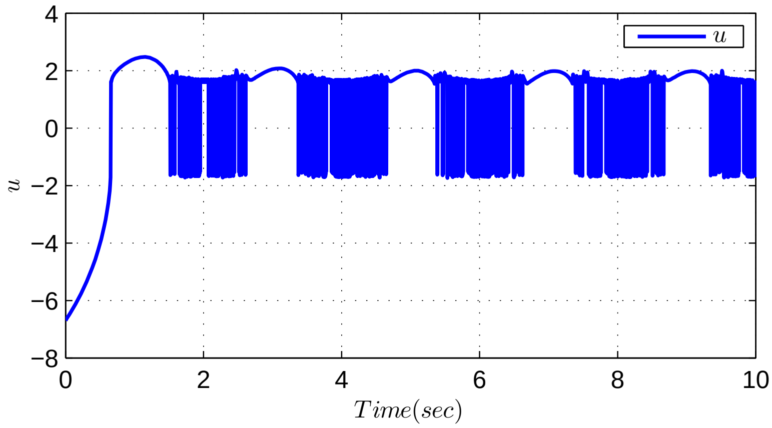

FIGURE 3.10: The control effort

We note from simulation results that the Super-Twisting Controller has all the properties which were present in the conventional sliding mode control with the extra remarkable property that now the control is smooth and the undesired high frequency chattering in the control action is now minimized.

C. Second Order Sliding Mode Control - The Smooth Super-Twisting Algorithm (SSTA):

The Super-Twisting Algorithm significantly attenuate the chattering phenomenon and maintains the accuracy of conventional sliding mode control. To smooth the control action the following second order sliding based controller called the Smooth

Super-Twisting Algorithm (SSTA) is considered for the example case.

$$\begin{aligned} u &= -\lambda_1 |\sigma|^{\frac{\rho-1}{\rho}} sign(\sigma) + w \\ \dot{w} &= -\lambda_2 |\sigma|^{\frac{\rho-2}{\rho}} sign(\sigma) \end{aligned} \tag{3.54}$$

where σ is the same sliding variable defined in (3.44) and λ_1, λ_2, and ρ are design constants for the controller gains. For the special case of $\rho = 2$, the SSTA becomes STA.

Simulation Results:

Fig. 3.11 - Fig. 3.13 show simulation results with controller parameters $\lambda_1 = 10$, $\lambda_2 = 15$, $\rho = 4$, $c = 1$, and system parameter $\beta = 1$ for initial condition $x_1(0) = 1$ and $x_2(0) = 2$.

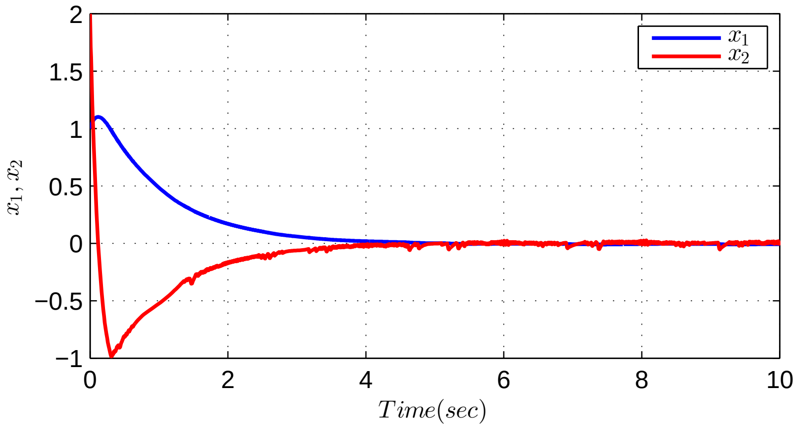

FIGURE 3.11: Closed loop response of system states for initial condition: $(x_1, x_2) = (1, 2)$)

We note that SMC, being first order, drives only σ to zero and $\dot{\sigma}$ oscillates between two constant values, $+0.707$ and -0.707 in this case. On the other hand, STA and SSTA both second order, drive both σ and $\dot{\sigma}$ to zero.

Example Summary:

1. *From the simulation results, we can deduce that sliding mode control techniques are robust against parametric uncertainties and external disturbances.*

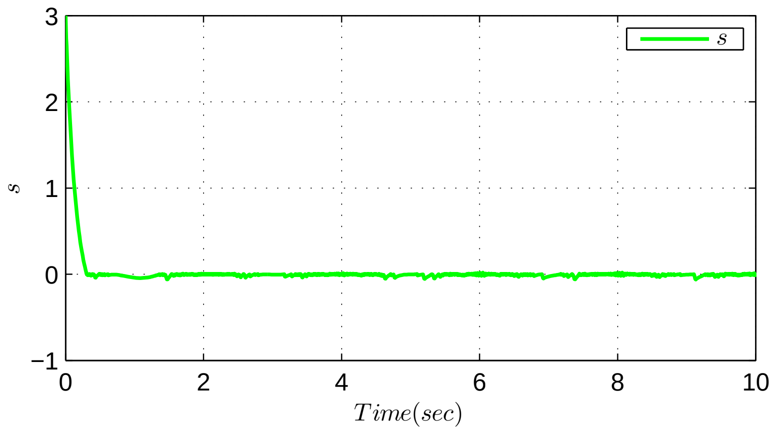

FIGURE 3.12: The sliding surface

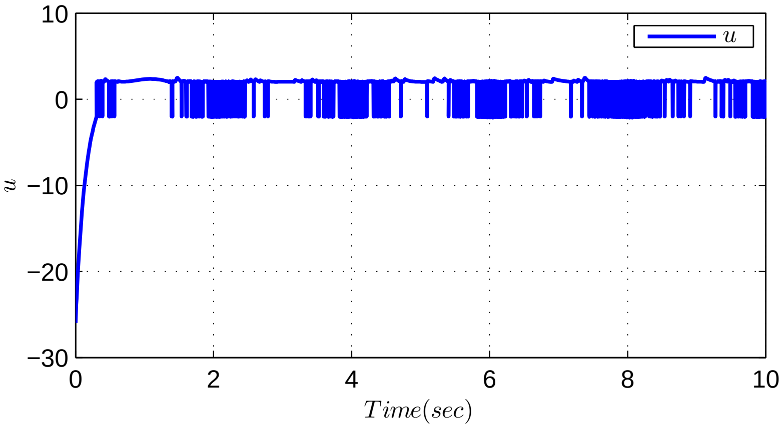

FIGURE 3.13: The control effort

2. *The undesired chattering phenomenon present in the conventional sliding mode control can be significantly attenuated with second-order STA and further smoothened by SSTA.*

3.6 Summary

This chapter provides an overview of the kinematic model of a nonholonomic system. The concept of backstepping, adaptive backstepping, sliding mode control, integral sliding mode control, second order and higher order sliding mode control techniques are presented in brief and coherent way to establish interest to utilize these techniques for the stabilization of nonholonomic systems. This chapter laid the foundation for the upcoming chapters of this dissertation.

Chapter 4

Stabilization of Nonholonomic Systems: Adaptive Backstepping Technique

4.1 Introduction

In the present chapter, we propose adaptive backstepping based design technique for nonholonomic systems with drift. The rigid body and the extended nonholonomic double integrator model belongs to this class of nonholonomic systems. The Brockett double integrator is considered to be a benchmark nonholonomic system. Research has shown that the extension of Brocket double integrator called the Extended Nonholonomic Double Integrator (ENDI) captures many properties of nonholomic systems with drift. See for instance, (Morgansen and Brockett [105]; Morgansen [106]; Aguiar et al. [107]; Aguiar and Pascoal [108]).

The kinematic model of nonholonomic systems with drift is represented as:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad x \in \Re^n \tag{4.1}$$

Our objective is to construct the stabilizing control algorithm for the class of systems represented by (4.1). The model of the rigid body and extended nonholonomic double integrator can be described as:

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2 \tag{4.2}$$

In order to stabilize system (4.2), an adaptive backstepping based controller is proposed, which yields asymptotic stabilization of the closed-loop system. This is achieved by first transforming the original system into a new system that can be made asymptotically stable. After the stabilization of the transformed system, the stability of the original system can be easily established. Numerical results show the effectiveness of the proposed control algorithm when compared to existing methods.

4.2 The Control Problem and Preliminaries

In the presence of suitable feedback strategy, a control law is designed such that as $t \to \infty$, $x(t;\ 0,\ x_0) \to x_{des}$ from any initial condition x_0. It is further supposed that $x_{des} = 0$ can be achieved by suitable transformation of the system.

4.3 The Proposed Control Algorithm

Consider the system:

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2$$

where

$$f(x) = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ 0 \\ 0 \\ h(x) \end{bmatrix}, \quad g_1(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \& \; g_2(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

which can be written as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-3} &= x_{n-2} \\ \dot{x}_{n-2} &= u_1 \\ \dot{x}_{n-1} &= u_2 \\ \dot{x}_n &= h(x) \end{aligned} \tag{4.3}$$

By choosing $u_1 = x_{n-1}$ and $u_2 = x_n + \theta\phi(x_1)$, the system (4.3) is written as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-3} &= x_{n-2} \\ \dot{x}_{n-2} &= x_{n-1} \\ \dot{x}_{n-1} &= x_n + \theta\phi(x_1) \\ \dot{x}_n &= h(x) \end{aligned} \tag{4.4}$$

Now with the help of below-mentioned steps, the system (4.4) can be transformed from $x \to z$ domain and then back from $z \to x$ by using inverse transformation.

Step 1: Firstly, the system (4.4) is transformed from the x to z domain with the help of time-varying transformation: $z = T(x, \theta\phi(x_1))$. Where θ is some unknown parameter estimated adaptively and $\phi(x_1)$ is some known function satisfying the condition $\phi(0) = 0$. Let $\hat{\theta}$ be the estimate of θ and $\tilde{\theta} = \theta - \hat{\theta}$ be the estimation error. Following are the properties of the proposed transformation:

i. $z = T(x, \theta\phi(x_1)) = Ax + B\hat{\theta}\phi(x_1)$, where $A \in \Re^{n\times n}$ and $B \in \Re^{n\times m}$ are some constant matrices.

ii. $x = \hat{T}(z, \theta\phi(x_1)) = \hat{A}z + \hat{B}\theta\phi(x_1)$, where, $A\hat{A} = I$ *i.e* $\hat{T}(z, \theta\phi(x_1)) = \text{inv}[T(x, \theta\phi(x_1))]$.

iii. $\dot{z} = Mz + N\tilde{\theta}\phi(x_1)$ is the transformed dynamical system, where M represents the negative definite matrix.

Step 2: Secondly, the adaptive law for $\tilde{\theta}$ is chosen in such a way, that yields asymptotic stabilization of the transformed system. The Lyapunov function is chosen as:

$$V(z, \tilde{\theta}) = \frac{1}{2}z^T z + \frac{1}{2}\tilde{\theta}^T\tilde{\theta}$$

Step 3: Finally, after the stabilization of the transformed system, the system (4.4) is stabilized at the origin.

$$x = \hat{T}(z, \theta\phi(x_1)) = \hat{A}z + \hat{B}\hat{\theta}\phi(x_1) \rightarrow 0 \text{ as } z \rightarrow 0\ ,\ \hat{\theta}\phi(\mathrm{x}_1) \rightarrow 0$$

Now the proposed algorithm is applied to two nonholonomic systems with drift: a rigid body and an extended nonholonomic double integrator system.

4.4 The Rigid Body

A rigid body in classical mechanics is an idealized model that starts with a non-deformable body. The body may have a continuous mass distribution, or it may be a system of discrete mass points (e.g., atoms, molecules). Non-deformability means that any two points of the body are always the same distance apart from external forces. Deformations such as deflection, compression, elongation or internal vibrations are thus excluded. The stabilization of such systems poses a challenging problem for researchers and scientists in the control domain.

4.4.1 Mathematical Model of the Rigid Body

The rigid body [109] has three states and two inputs and it mathematical model can be written as:

$$\dot{x} = f(x) + \sum_{i=1}^{2} g_i(x)u_i \tag{4.5}$$

where $x \in \Re^3$ is the state vector and $u \in \Re^2$ is the control input vector.

$$f(x) = \begin{bmatrix} J_{23}x_2x_3 \\ J_{31}x_3x_1 \\ J_{12}x_1x_2 \end{bmatrix}, \; g_1(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; g_2(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{4.6}$$

Also $J_{23} = \frac{J_2 - J_3}{J_1}$, $J_{31} = \frac{J_3 - J_1}{J_2}$, $J_{12} = \frac{J_1 - J_2}{J_3}$.

where J_1, J_2 and J_3 are the primary moment of inertia respectively.

For controllability, the Lie Algebra Rank Condition (LARC) condition has to be satisfied. For this, we compute:

$$g_3(x) = [f, g_1](x) = \begin{bmatrix} 0 \\ -J_{31}x_3 \\ -J_{12}x_2 \end{bmatrix}$$

$$g_4(x) = [f, g_2](x) = \begin{bmatrix} -J_{23}x_3 \\ 0 \\ -J_{12}x_1 \end{bmatrix}$$

$$g_5(x) = [f, g_4](x) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

we conclude that LARC condition is satisfied. $span\{g_1, g_2, g_5\}(x) = \Re^3 \ \forall \ x \in \mathbb{R}^3$, so the system is controllable. The rigid body model (4.6) can be expressed as:

$$\dot{x}_1 = J_{23}x_2x_3 + u_1 \tag{4.7a}$$

$$\dot{x}_2 = J_{13}x_1x_3 + u_2 \tag{4.7b}$$

$$\dot{x}_3 = J_{12}x_1x_2 \tag{4.7c}$$

4.4.2 Construction of the Transformation

By above system, let

$$\begin{aligned} u_1 &= x_2 - J_{23}x_2x_3 \\ u_2 &= x_3 - J_{31}x_3x_1 + \theta\phi(x_1) \end{aligned} \tag{4.8}$$

Then the above system becomes:

$$\dot{x}_1 = x_2 \tag{4.9a}$$

$$\dot{x}_2 = x_3 + \hat{\theta}\phi(x_1) + \tilde{\theta}\phi(x_1) \tag{4.9b}$$

$$\dot{x}_3 = J_{12}x_1x_2 \tag{4.9c}$$

Consider Eq. (4.9a): $\dot{x}_1 = x_2$. Consider x_2 state as a virtual input and α_1 as a stabilizing function, then the error variable $z_1 = x_2 - \alpha_1$. Eq. (4.9a) becomes:

$$\dot{x}_1 = z_1 + \alpha_1 \tag{4.10}$$

Consider the Lyapunov function $V_0 = \frac{1}{2}x_1^2$ for computing the stabilizing function α_1. Then:

$$\dot{V}_0 = x_1\dot{x}_1 = x_1(z_1 + \alpha_1)$$

Choose $\alpha_1 = -x_1$, then $\dot{V}_0$ becomes:

$$\dot{V}_0 = -x_1^2 + x_1 z_1 \tag{4.11}$$

The term $x_1 z_1$ will be canceled in the next step. Eq. (4.9a) becomes:

$$\dot{x}_1 = z_1 - x_1 \tag{4.12}$$

Consider (4.9b) as: $\dot{x}_2 = x_3 + \hat{\theta}\phi(x_1) + \tilde{\theta}\phi(x_1)$. Consider x_3 state as a virtual input and α_2 as a stabilizing function, then the error variable is: $z_2 = x_3 - \alpha_2$. Eq. (4.9b) becomes:

$$\dot{x}_2 = z_2 + \alpha_2 + \hat{\theta}\phi(x_1) + \tilde{\theta}\phi(x_1) \tag{4.13}$$

and $z_1 = x_2 - \alpha_1 = x_2 + x_1 \Rightarrow \dot{z}_1 = \dot{x}_2 + \dot{x}_1$

$$\dot{z}_1 = z_2 + \alpha_2 + z_1 - x_1 + \hat{\theta}\phi(x_1) + \tilde{\theta}\phi(x_1) \tag{4.14}$$

Consider the Lyapunov function $V_1 = V_0 + \frac{1}{2}z_1^2$ for computing the stabilizing function α_2. Then:

$$\begin{aligned}\dot{V}_1 = \dot{V}_0 + z_1\dot{z}_1 &= x_1 z_1 - {x_1}^2 + z_1\{z_2 + \alpha_2 + \hat{\theta}\phi(x_1) + \tilde{\theta}\phi(x_1) + z_1 - x_1\} \\ &= -{x_1}^2 + z_1\{z_2 + x_2 + \hat{\theta}\phi(x_1) + z_1\} + \tilde{\theta}\phi(x_1)z_1 \end{aligned} \tag{4.15}$$

Choose $\alpha_2 = -2z_1 - \hat{\theta}\phi(x_1)$, then $\dot{V}_1 = -x_1^2 + z_1 z_2 - z_1^2 + \tilde{\theta}\phi(x_1)z_1$. The term $z_1 z_2$ will be canceled in the next step. Eq. (4.9c) becomes:

$$\dot{z}_1 = z_2 - z_1 - x_1 + \tilde{\theta}\phi(x_1) \tag{4.16}$$

The stabilizing function α_2 can be written as:

$$\alpha_2 = -2z_1 - \hat{\theta}\phi(x_1) = -2z_1 - v + \hat{\theta}$$

where $v = \hat{\theta}(\phi(x_1) + 1)$ and $z_2 = x_3 - \alpha_2 = x_3 + 2z_1 + v - \hat{\theta}$

$$\dot{z}_2 = \dot{x}_3 + 2\dot{z}_1 + \dot{v} - \dot{\hat{\theta}} = J_{12}x_1x_2 + 2z_2 - 2z_1 - 2x_1 - 2\tilde{\theta}\phi(x_1) + \dot{v} - \dot{\hat{\theta}} \quad (4.17)$$

Consider the Lyapunov function:

$$V_2 = V_1 + \frac{1}{2}z_2^2 + \frac{1}{2}\tilde{\theta}2 \quad (4.18)$$

$$\dot{V}_2 = \dot{V}_1 + z_2\dot{z}_2 + \tilde{\theta}\dot{\tilde{\theta}} = -x_1^2 + z_1z_2 - z_1^2 + z_2\{J_{12}x_1x_2 + 2z_2 - 2z_1 - 2x_1 + \dot{v} - \dot{\hat{\theta}}\}$$
$$+ \tilde{\theta}\{(z_1 - 2z_2)\phi(x_1) + \dot{\tilde{\theta}}\} \quad (4.19)$$

Choose $\dot{v} = -x_1x_2 - 3z_2 + z_1 + 2x_1 + \dot{\hat{\theta}}$, and $\dot{\tilde{\theta}} = (2z_2 - z_1)\phi(x_1) - \tilde{\theta} = -\dot{\hat{\theta}}$.

Eq. (4.19) becomes:

$$\dot{V}_2 = -x_1^2 - z_1^2 - z_2^2 - \tilde{\theta}^2$$

Eq. (4.17) becomes:

$$\dot{z}_2 = -z_2 - z_1 + 2\tilde{\theta}\phi(x_1)$$

So the response of system is written as:

$$\begin{aligned}
\dot{x}_1 &= z_1 - x_1 \\
\dot{z}_1 &= z_2 - z_1 - x_1 + \tilde{\theta}\phi(x_1) \\
\dot{z}_2 &= -z_2 - z_1 + 2\tilde{\theta}\phi(x_1) \\
\dot{\tilde{\theta}} &= (2z_2 - z_1)\phi(x_1) - \tilde{\theta}
\end{aligned} \quad (4.20)$$

Choose overall Lyapunov function as:

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2}\tilde{\theta}^2 \quad (4.21)$$

$$\dot{V} = x_1(\dot{x}_1) + z_1(\dot{z}_1) + z_2(\dot{z}_2) + \tilde{\theta}\dot{\tilde{\theta}}$$

$$\begin{aligned}\dot{V} &= x_1(z_1 - x_1) + z_1(z_2 - z_1 - x_1 + \tilde{\theta}\phi(x_1)) + z_2(-z_2 - z_1 + 2\tilde{\theta}\phi(x_1)) \\ &+ \tilde{\theta}\{(2z_2 - z_1)\phi(x_1) - \tilde{\theta}\}\end{aligned} \tag{4.22}$$

Simplifying above equation, we get

$$\dot{V} = -x_1^2 - z_1^2 - z_2^2 - \tilde{\theta}^2 \tag{4.23}$$

Define: $z = [x_1,\, z_1,\, z_2]^T$

$$\dot{z} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \tilde{\theta}\phi(x_1) = Mz + N\tilde{\theta}\phi(x_1) \tag{4.24}$$

where M is negative definite. Since the derivative of Lyapunov function given by (4.23) is negative, therefore, we conclude that the transformed system (4.24) is asymptotically stable, therefore, it implies that $x_1, z_1, z_2 \to 0$ and $\tilde{\theta} \to 0$.

By using the relations:

$$\begin{aligned} x_1 &= x_1 \\ z_1 &= x_1 + x_2 \\ z_2 &= x_3 + 2z_1 + \hat{\theta}\phi(x_1) = x_3 + 2x_1 + 2x_2 + \hat{\theta}\phi(x_1) \end{aligned} \tag{4.25}$$

we can write:

$$z = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \hat{\theta}\phi(x_1) = Ax + B(\hat{\theta}\phi(x_1)) = T(x, \theta\phi(x_1)) \tag{4.26}$$

where, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Also using the following relations:

$$\begin{aligned} x_1 &= x_1 \\ x_2 &= z_1 - x_1 \\ x_3 &= z_2 - 2z_1 - \hat{\theta}\phi(x_1) \end{aligned} \tag{4.27}$$

$$x = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \hat{\theta}\phi(x_1) = \hat{A}z + \hat{B}(\hat{\theta}\phi(x_1)) = \hat{T}(z, \theta\phi(x_1)) \tag{4.28}$$

From above, $A\hat{A} = I$ & $\hat{A}B = \hat{B}$. As $z \to 0$ & $\hat{\theta}\phi(x_1) \to 0$, so $x \to 0$, implies $x_2 = z_1 - x_1 \to 0$ & $x_3 = z_2 - 2z_1 - \hat{\theta}\phi(x_1) \to 0$.

4.4.3 Simulation Results

The closed loop systems becomes:

$$\begin{aligned} \dot{x}_1 &= z_1 - x_1 \\ \dot{z}_1 &= z_2 - z_1 - x_1 + \tilde{\theta}\phi(x_1) \\ \dot{z}_2 &= -z_2 - z_1 + 2\tilde{\theta}\phi(x_1) \\ \dot{\tilde{\theta}} &= (2z_2 - z_1)\phi(x_1) - \tilde{\theta} \end{aligned} \tag{4.29}$$

The closed loop system is now simulated using MATLAB. The simulation results with initial are: $(x_1, z_1, z_2) = (0.2, 0.5, 0.8)$. Our objective of stabilizing the system from any initial state to any desired value has been achieved as shown by the results.

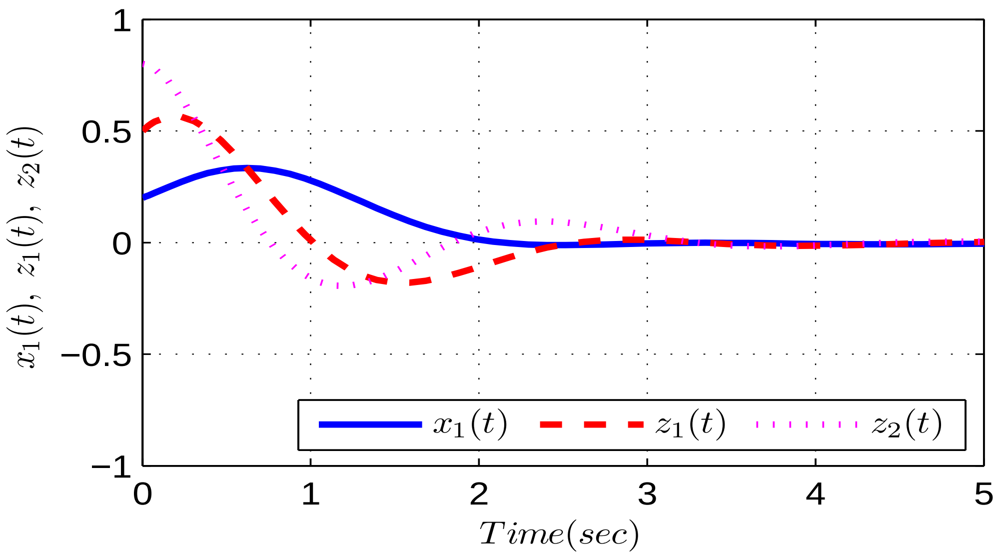

FIGURE 4.1: Closed loop response of the rigid body for initial condition: $[x_1, z_1, z_2] = [0.2, 0.5, 0.8]^T$.

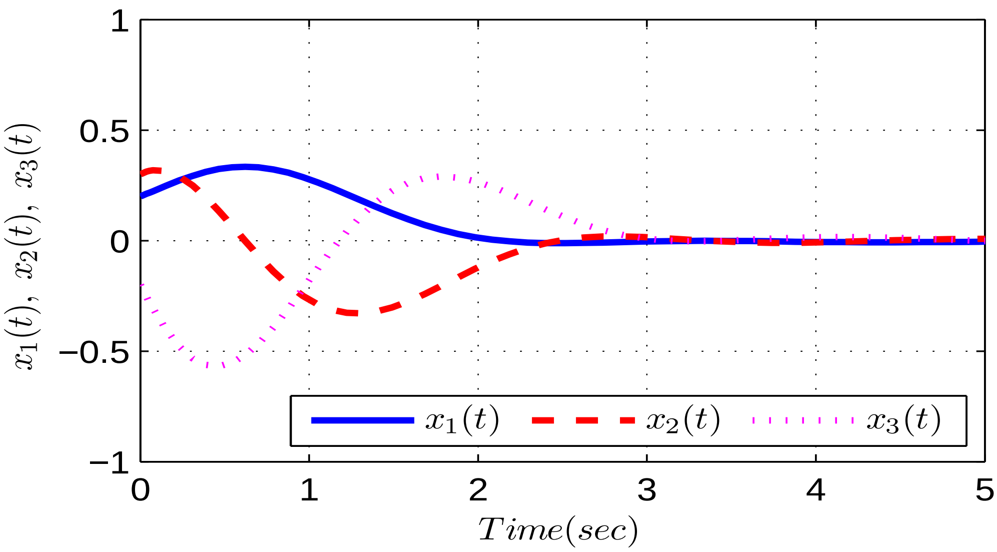

FIGURE 4.2: Closed loop response of the rigid body for initial condition: $[x_1, x_2, x_3] = [0.2, 0.3, -0.2]^T$.

4.5 Extended Nonholonomic Double Integrator

The nonholonomic integrator has been widely studied and discussed due to its structure which is quite related to many electromechanical systems like induction motors and mobile robots. The control of such system is also of interest as such systems are not stabilizable by smooth static feedback laws (Brockett [1]). The work in literature for the stabilization and tracking control of such systems is mainly based on the nonsmooth feedback laws (Bloch et al. [110, 111], Khennouf and de Wit [112]), or smooth time-varying feedback laws (Pomet [3], M'Closkey and Murray [113]). The work by Dixon et al. [114] is based on the design of control laws which stabilize both current-fed induction motor and the double integrator system.

The nonholonomic integrator can be generalized by adding some cascade integrator before the input and perturbation terms in all its equations. The application area of this extended nonholonomic double integrator may be voltage fed induction motors or mobile robots. In fact, the stabilization and tracking control of nonholonomic integrator or extended nonholonomic double integrator led to the solution of many electromechanical nonholonomic systems.

4.5.1 Mathematical Model of an Extended Nonholonomic Double Integrator

The nonholonomic integrator is represented as:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 - x_1 u_2 \end{aligned} \tag{4.30}$$

Where $x \in \Re^3$ is the state vector and $u \in \Re^2$ is the control input vector. The above-mentioned system has all the properties of nonholonomic systems and is known as a benchmark system for nonholonomic control system design and analysis in the literature [1, 2, 5].

The kinematic model of nonholonomic integrator (4.30) exhibits all the properties of nonholonomic systems like wheel mobile robots. However, when we take into account both the kinematics and dynamics of the wheel robots, the nonholonomic integrator system fails to capture all the features. To represent a more realistic case, we must use the extended nonholonomic double integrator model. The equations of motion of an extended nonholonomic double integrator can be represented into the following form [1]:

$$\begin{aligned} \ddot{x}_1 &= u_1 \\ \ddot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1\dot{x}_2 - x_2\dot{x}_1 \end{aligned} \tag{4.31}$$

where state variable $x = [x_1, x_2, x_3, x_4, x_5]^T = [x_1, x_2, x_3, \dot{x}_1, \dot{x}_2]^T$, system (4.31) can be written as:

$$\begin{aligned} \dot{x}_1 &= x_4 \\ \dot{x}_2 &= x_5 \\ \dot{x}_3 &= x_1x_5 - x_2x_4 \\ \dot{x}_4 &= u_1 \\ \dot{x}_5 &= u_2 \end{aligned} \tag{4.32}$$

The above system (4.32) is rewritten in general way as:

$$\dot{x} = f_0(x) + g_1(x)u_1 + g_2(x)u_2 \tag{4.33}$$

where

$$f_0(x) = \begin{bmatrix} x_4 \\ x_5 \\ x_1x_5 - x_2x_4 \\ 0 \\ 0 \end{bmatrix}, \; g_1(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The system (4.33) is referred as the Extended Nonholonomic Double Integrator (ENDI). The ENDI system (4.33) must fulfill the following characteristic:

i. The vector f_0, g_1, g_2 are complete real and analytic, Furthermore, $f_0(0) = 0$.

ii. The Extended nonholonomic double integrator is accessible for $x \in \Re^5$, as Lie algebra rank condition (LARC) is been satisfied: $span\{g_1, g_2, g_3, g_4, g_5\} = R^5,\ \forall\, x \in R^5$.

where

$$g_3(x) = [f_0(x), g_1(x)] = \begin{bmatrix} 1 \\ 0 \\ -x_2 \\ 0 \\ 0 \end{bmatrix} \tag{4.34a}$$

$$g_4(x) = [f_0(x), g_2(x)] = \begin{bmatrix} 0 \\ 1 \\ x_1 \\ 0 \\ 0 \end{bmatrix} \tag{4.34b}$$

$$g_5(x) = [g_3(x), g_4(x)] = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \tag{4.34c}$$

4.5.2 Construction of the Transformation

By choosing $u_1 = x_2,\ u_2 = x_3 + \theta\phi(x_1)$, system (4.32) can be written as:

$$\dot{x}_1 = x_4 \tag{4.35a}$$

$$\dot{x}_4 = x_2 \tag{4.35b}$$

$$\dot{x}_2 = x_5 \tag{4.35c}$$

$$\dot{x}_5 = x_3 + \hat{\theta}\phi(x_1) + \tilde{\theta}\phi(x_1) \tag{4.35d}$$

$$\dot{x}_3 = x_1 x_5 - x_2 x_4 \tag{4.35e}$$

In Eq. (4.35a), consider x_4 as virtual control and α_1 as the stabilizing function, then the error variable becomes: $z_1 = x_4 - \alpha_1$, Eq. (4.35a) becomes:

$$\dot{x}_1 = z_1 + \alpha_1 \tag{4.36}$$

By considering the Lyapunov Function $V_0 = \frac{1}{2}x_1^2$ for (4.36), the derivative of V_0 becomes:

$$\dot{V}_0 = x_1\dot{x}_1 = x_1(z_1 + \alpha_1) \tag{4.37}$$

By taking $\alpha_1 = -x_1$, Eq. (4.37) becomes:

$$V_0 = -x_1^2 + x_1 z_1 \tag{4.38}$$

and Eq. (4.36) can be written as:

$$\dot{x}_1 = z_1 - x_1 \tag{4.39}$$

Now consider Eq. (4.35b) and take x_2 as virtual control input and α_2 as the stabilizing function, then the error variable becomes: $z_2 = x_2 - \alpha_2$, Eq. (4.35b) becomes:

$$\dot{x}_4 = z_2 + \alpha_2 \tag{4.40}$$

Since the dynamic of first error variable $z_1 = x_4 - \alpha_1 = x_4 + x_1$ is:

$$\dot{z}_1 = \dot{x}_4 + \dot{x}_1 = z_2 + \alpha_2 + z_1 - x_1 \tag{4.41}$$

By considering the Lyapunov function: $V_1 = V_0 + \frac{1}{2}z_1^2$ for (4.39) and (4.41), the derivative of V_0 becomes:

$$\dot{V}_1 = -x_1^2 + z_1\{z_2 + \alpha_2 + z_1\} \tag{4.42}$$

Choose $\alpha_2 = -2z_1$

$$\dot{V}_1 = -x_1^2 - z_1^2 + z_1 z_2 \tag{4.43}$$

The above Eq. (4.41) can be written as:

$$\dot{z}_1 = z_2 - z_1 - x_1 \tag{4.44}$$

By considering Eq. (4.35c) and taking x_5 as virtual control and α_3 as a stabilizing function, the error variable can be written as: $z_3 = x_5 - \alpha_3$, Eq. (4.35c) becomes:

$$\dot{x}_2 = x_5 = z_3 + \alpha_3 \tag{4.45}$$

The dynamic of the second error variable namely $z_2 = x_2 - \alpha_2 = x_2 + 2z_1$ is:

$$\dot{z}_2 = \dot{x}_2 + 2\dot{z}_1 = z_3 + \alpha_3 + 2z_2 - 2z_1 - 2x_1 \tag{4.46}$$

To compute α_3, choose Lyapunov function as: $V_2 = V_1 + \frac{1}{2}z_2^2$ for (4.39), (4.44) and (4.46). Then the derivative of the Lyapunov is written as:

$$\dot{V}_2 = -x_1^2 - z_1^2 + z_2\{z_3 + \alpha_3 + 2z_2 - z_1 - 2x_1\} \tag{4.47}$$

Choose $\alpha_3 = -3z_2 + z_1 + 2x_1$, so the derivative of the V_2 becomes:

$$\dot{V}_2 = -x_1^2 - z_1^2 - z_2^2 + z_2 z_3 \tag{4.48}$$

Eq. (4.46) becomes:

$$\dot{z}_2 = z_3 - z_2 - z_1 \tag{4.49}$$

In Eq. (4.35d), consider x_3 state as a virtual input, and α_4 as a stabilizing function and choose $z_4 = x_3 - \alpha_4$ be the error function, Eq. (4.35d) can be written as:

$$\dot{x}_5 = x_3 + \hat{\theta}\phi(x_1) + \tilde{\theta}\phi(x_1) = z_4 + \alpha_4 + \hat{\theta}\phi(x_1) + \tilde{\theta}\phi(x_1) \tag{4.50}$$

The dynamic of third error variable namely $z_3 = x_5 - \alpha_3 = x_5 + 3z_2 - z_1 - 2x_1$ is:

$$\begin{aligned}\dot{z}_3 &= \dot{x}_5 + 3\dot{z}_2 - \dot{z}_1 - 2\dot{x}_1\\ &= z_4 + \alpha_4 + \hat{\theta}\phi(x_1) + \tilde{\theta}\phi(x_1) + 3z_3 - 3z_2 - 3z_1 - z_2\\ &+ z_1 + x_1 - 2z_1 + 2x_1\\ &= z_4 + \alpha_4 + 3z_3 - 4z_2 - 4z_1 + 3x_1 + \hat{\theta}\phi(x_1) + \tilde{\theta}\phi(x_1)\end{aligned} \tag{4.51}$$

For α_4, choose the Lyapunov function as: $V_3 = V_2 + \frac{1}{2}z_3^2$ for (4.39), (4.44), (4.49) and (4.51). Then

$$\begin{aligned}\dot{V}_3 &= \dot{V}_2 + z_3\dot{z}_3\\ &= -x_1^2 - z_1^2 - z_2^2 + z_3\{z_4 + \alpha_4 + 3z_3 - 3z_2 - 4z_1 + 3x_1 + \hat{\theta}\phi(x_1)\}\\ &+ z_3\tilde{\theta}\phi(x_1)\end{aligned} \tag{4.52}$$

By choosing $\alpha_4 = -4z_3 + 3z_2 + 4z_1 - 3x_1 - \hat{\theta}\phi(x_1)$, we have

$$\dot{V}_3 = -x_1^2 - z_1^2 - z_2^2 - z_3^2 + z_3z_4 + \tilde{\theta}\phi(x_1)z_3 \tag{4.53}$$

Eq. (4.51) becomes:

$$\dot{z}_3 = z_4 - z_3 - z_2 + \tilde{\theta}\phi(x_1) \tag{4.54}$$

The stabilizing function α_4 can be written as:

$$\begin{aligned}\alpha_4 &= -4z_3 + 3z_2 + 4z_1 - 3x_1 - \hat{\theta}\phi(x_1)\\ &= -4z_3 + 3z_2 + 4z_1 - 3x_1 - v + \hat{\theta}\end{aligned} \tag{4.55}$$

where $v = \hat{\theta}(\phi(x_1) + 1)$.

The dynamics of fourth error variable namely $z_4 = x_3 - \alpha_4 = x_3 + 4z_3 - 3z_2 - 4z_1 + 3x_1 + v - \hat{\theta}$ is:

$$\begin{aligned}
\dot{z}_4 &= \dot{x}_3 + 4\dot{z}_3 - 3\dot{z}_2 - 4\dot{z}_1 + 3\dot{x}_1 + \dot{v} - \dot{\hat{\theta}} \\
&= x_1x_5 - x_2x_4 + 4z_4 - 4z_3 - 4z_2 + 4\tilde{\theta}\phi(x_1) - 3z_3 \\
&+ 3z_2 + 3z_1 - 4z_2 + 4z_1 + 4x_1 + 3z_1 - 3x_1 + \dot{v} - \dot{\hat{\theta}} \\
&= \beta + 4z_4 - 7z_3 - 5z_2 + 10z_1 + x_1 + 4\tilde{\theta}\phi(x_1) + \dot{v} - \dot{\hat{\theta}}
\end{aligned} \tag{4.56}$$

where

$$\beta = x_1x_5 - x_2x_4 = x_1(z_3 - 3z_2 + z_1 + 2x_1) - (z_2 - 2z_1)(z_1 - x_1) \tag{4.57}$$

Consider the Lyapunov function $V_4 = V_3 + \frac{1}{2}z_4^2 + \tilde{\theta}^2$ for (4.39), (4.44), (4.49), (4.54) and (4.56).

$$\begin{aligned}
\dot{V}_4 &= \dot{V}_3 + z_4\dot{z}_4 + 2\tilde{\theta}\dot{\tilde{\theta}} \\
&= -x_1^2 - z_1^2 - z_2^2 - z_3^2 + z_4\{\beta + 4z_4 - 6z_3 - 5z_2 + 10z_1 + x_1 + \dot{v} - \dot{\hat{\theta}}\} \\
&+ \tilde{\theta}\{(z_3 + 4z_4)\phi(x_1) + \dot{\tilde{\theta}}\}
\end{aligned} \tag{4.58}$$

By choosing

$$\dot{v} = -\beta - 5z_4 + 6z_3 + 5z_2 - 10z_1 - x_1 + \dot{\hat{\theta}} \tag{4.59}$$

$$\dot{\tilde{\theta}} = -(z_3 + 4z_4)\phi(x_1) - \tilde{\theta} = -\dot{\hat{\theta}} \tag{4.60}$$

we have

$$\dot{V}_4 = -{x_1}^2 - {z_1}^2 - {z_2}^2 - {z_3}^2 - {z_4}^2 - \tilde{\theta}^2 \tag{4.61}$$

and the Eq. (4.56) become:

$$\dot{z}_4 = -z_4 - z_3 + 4\tilde{\theta}\phi(x_1) \tag{4.62}$$

The closed loop response becomes:

$$\begin{aligned}
\dot{x}_1 &= z_1 - x_1 \\
\dot{z}_1 &= z_2 - z_1 - x_1 \\
\dot{z}_2 &= z_3 - z_2 - z_1 \\
\dot{z}_3 &= z_4 - z_3 - z_2 + \tilde{\theta}\phi(x_1) \\
\dot{z}_4 &= -z_4 - z_3 + 4\tilde{\theta}\phi(x_1)
\end{aligned} \tag{4.63}$$

By defining state variable as: $z = [x_1, z_1, z_2, z_3, z_4]^T$, we have:

$$\dot{z} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} \tilde{\theta}\phi(x_1) \end{bmatrix}$$

$$= Mz + N\tilde{\theta}\phi(x_1) \tag{4.64}$$

It can be easily verified that M is negative definite. Since the derivative of Lyapunov function given by (4.61) is strictly negative, therefore, the transformed system (4.64) is asymptotically stable, therefore, $x_1, z_1, z_2, z_3, z_4 \to 0$ and $\tilde{\theta} \to 0$.

By using the following relations

$$\begin{aligned}
x_1 &= x_1 \\
z_1 &= x_4 - \alpha_1 = x_4 + x_1 \\
z_2 &= x_2 - \alpha_2 = x_2 + 2x_1 + 2x_4 \\
z_3 &= x_5 + 3z_2 - z_1 - 2x_1 = x_5 + 5x_4 + 3x_2 + 3x_1 \\
z_4 &= x_3 - \alpha_4 = x_3 + 4z_3 - 3z_2 - 4z_1 + 3x_1 + \hat{\theta}\phi(x_1) \\
&= 4x_5 + 10x_4 + x_3 + 9x_2 + 5x_1 + \hat{\theta}\phi(x_1)
\end{aligned} \tag{4.65}$$

we can write:

$$z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 & 0 \\ 3 & 3 & 0 & 5 & 1 \\ 5 & 9 & 1 & 10 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \hat{\theta}\phi(x_1) \end{bmatrix}$$

$$= Ax + B\hat{\theta}\phi(x_1) = T(x, \theta\phi(x_1)) \tag{4.66}$$

Also using the following relations

$$\begin{aligned} x_1 &= x_1 \\ x_4 &= z_1 - x_1 \\ x_2 &= z_2 - 2z_1 \\ x_5 &= z_3 - 3z_2 + z_1 + 2x_1 \\ x_3 &= z_4 - 4z_3 + 3z_2 + 4z_1 - 3x_1 - \hat{\theta}\phi(x_1) \end{aligned} \tag{4.67}$$

we have:

$$x = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ -3 & 4 & 3 & -4 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 2 & 1 & -3 & 1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \hat{\theta}\phi(x_1) \end{bmatrix}$$

$$= \hat{A}z + \hat{B}\hat{\theta}\phi(x_1) = \hat{T}(z, \theta\phi(x_1)) \tag{4.68}$$

From above we can easily verified that $A\hat{A} = I$ and $\hat{A}B = -\hat{B}$. As $z \to 0$ and $\hat{\theta}\phi(x_1) \to 0$, so $x \to 0$, thus the original system (4.68) will also converge asymptotically.

Theorem 4.1. *By Choosing* $u_1 = x_2$ *and* $u_2 = x_3 + \theta\phi(x_1)$ *and the transformation* *as given in (4.66) the system (4.68) can be transformed into (4.66) which is asymptotically stable and therefore, by (4.68) the original system is always asymptotically stable*

Proof. By choosing a Lyapunov function as: $V(z) = \frac{1}{2}z^T z + \frac{1}{2}\tilde{\theta}^2$, the derivative along the system trajectories becomes:

$$\begin{aligned}\dot{V}_z &= x_1\dot{x}_1 + z_1\dot{z}_1 + z_2\dot{z}_2 + z_3\dot{z}_3 + z_4\dot{z}_4 + \tilde{\theta}\dot{\tilde{\theta}} \\ &= x_1(z_1 - x_1) + z_1(z_2 - z_1 - x_1) + z_2(z_3 - z_2 - z_1) + z_3(z_4 - z_3 - z_2 + \tilde{\theta}\phi(x_1)) \\ &+ z_4(-z_4 - z_3 + 4\tilde{\theta}\phi(x_1)) + \tilde{\theta}\dot{\tilde{\theta}} \\ &= -x_1^2 - z_1^2 - z_2^2 - z_3^2 - z_4^2 + \tilde{\theta}\{\dot{\tilde{\theta}} + z_3\phi(x_1) + 4z_4\phi(x_1)\} \\ &= -x_1^2 - z_1^2 - z_2^2 - z_3^2 - z_4^2 - \tilde{\theta}^2 < 0\end{aligned} \tag{4.69}$$

By choosing the Lyapunov function as:

$$V(z) = \frac{1}{2}z^T z + \frac{1}{2}\tilde{\theta}^2 \tag{4.70}$$

and the adaptive law as:

$$\dot{\tilde{\theta}} = -z_3\phi(x_1) - 4z_4\phi(x_1) - \tilde{\theta} \tag{4.71}$$

we will come up with the asymptotic stability of Lyapunov function:

$$\dot{V} < 0$$

From the above equation we conclude the asymptotically stability of the transformed system and by (4.68) the system (4.35a)-(4.35e) is also asymptotically stable. □

4.5.3 Simulation Results

Results show the effectiveness of the proposed method in stabilizing the ENDI system. The aim of the control design is to stabilize the states of the system to the origin. Fig. 4.3 - Fig. 4.6 show simulation results for different initial conditions. Fig. 4.7 - Fig. 4.11 shows the effect of k on the transient response of the system for different values of k. The results of the proposed controller are compared to those of [115] in which a continuous, and a discontinuous controller are presented. The initial conditions are chosen to be the same as $x = [0.9, 0.7, 0.4, 0.8, 0.6]$ for each case. Fig. 4.12 shows the result of the proposed algorithm. Fig. 4.13 and the Fig. 4.14 shows the results of the continuous and discontinuous controller respectively. The comparison shows that the response of the proposed controller is better than that of [115] regarding settling time and is less oscillatory.

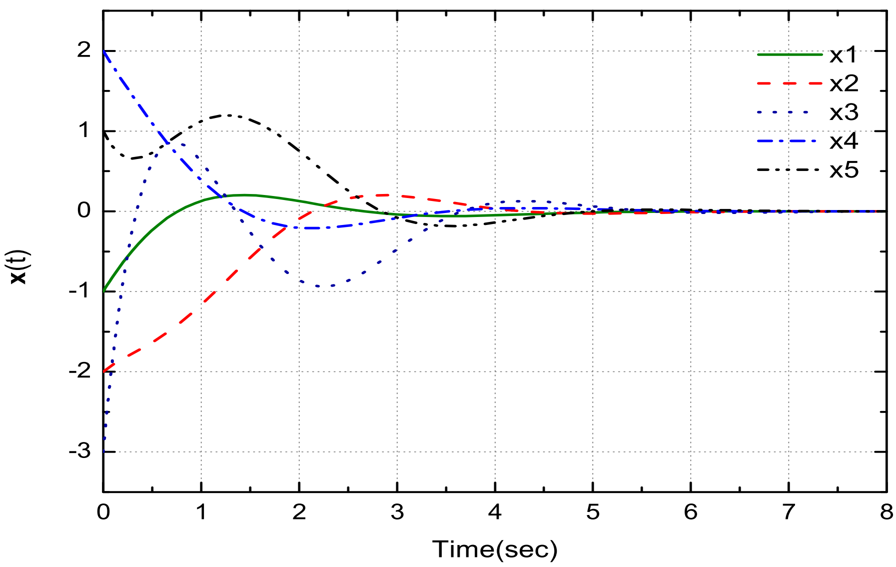

FIGURE 4.3: Closed loop response of the extended nonholonomic double integrator for initial condition:
$x = [-1, -2, -3, 2, 1]^T$.

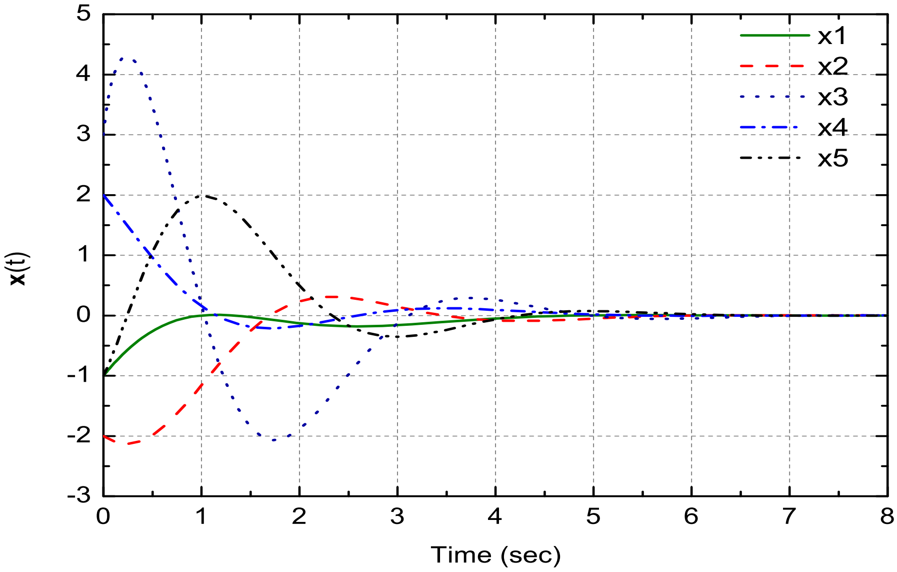

FIGURE 4.4: Closed loop response of the extended nonholonomic double integrator for initial condition:
$x = [-1, -2, 3, 2, -1]^T$.

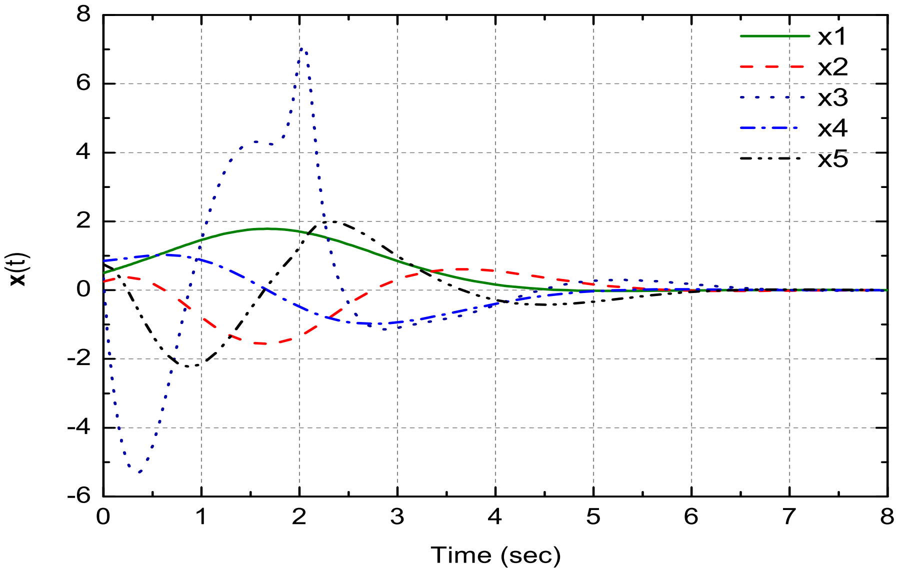

FIGURE 4.5: Closed loop response of the extended nonholonomic double integrator for initial condition:
$x = [.5, .25, .35, .85, .75]^T$.

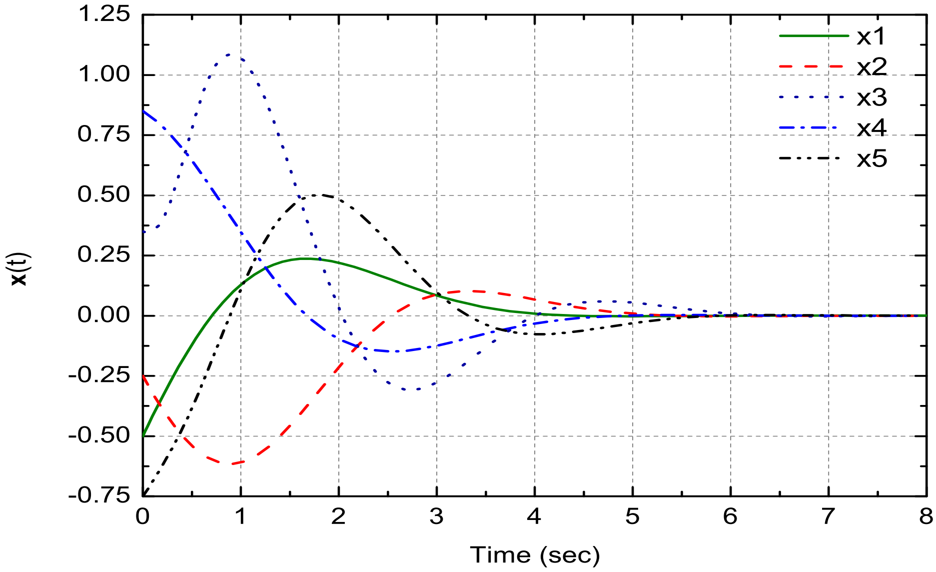

FIGURE 4.6: Closed loop response of the extended nonholonomic double integrator for initial condition:
$x = [-.5, -.25, .35, .85, -.75]^T$

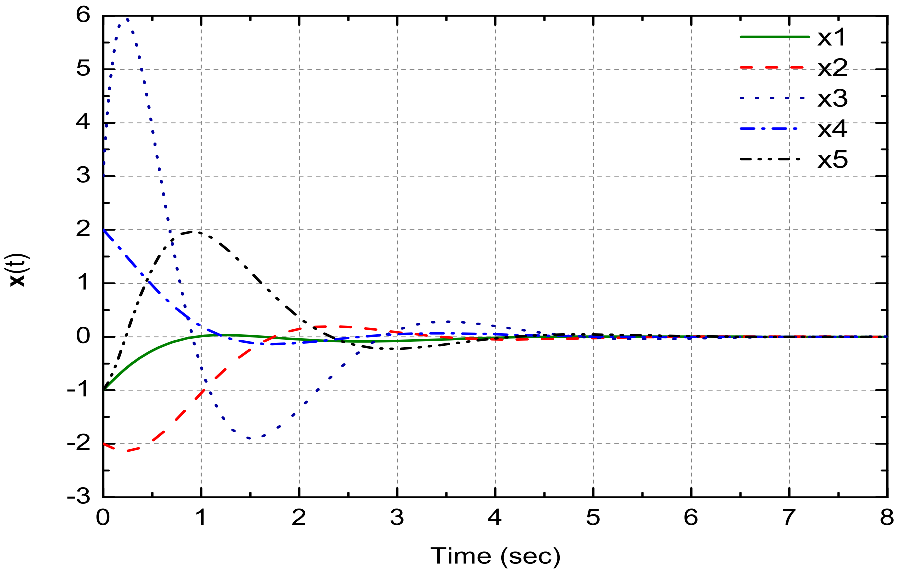

FIGURE 4.7: Closed loop response of the extended nonholonomic double integrator for initial condition:
$x = [-1, -2, 3, 2, -1]^T$.

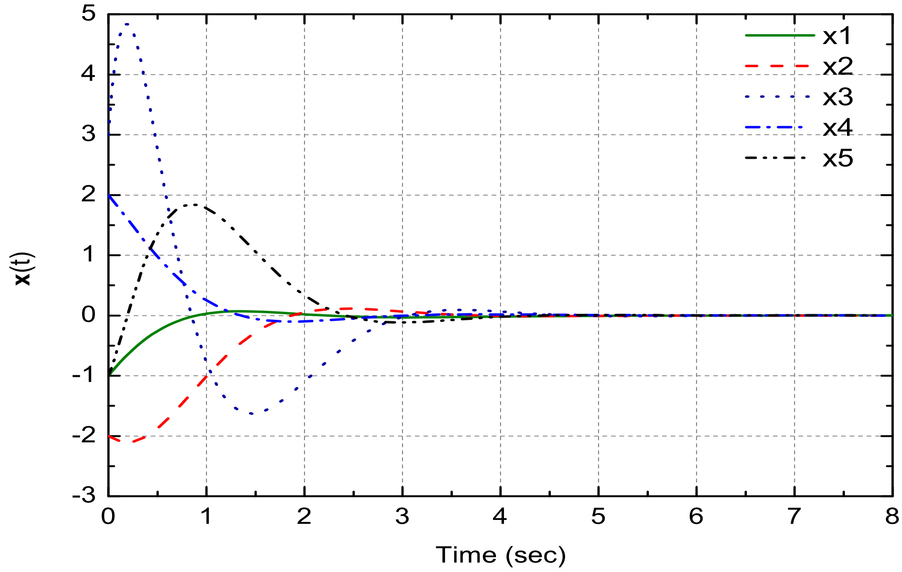

FIGURE 4.8: Closed loop response of the extended nonholonomic double integrator with gain $k = -1$ and initial condition: $x = [-1, -2, 3, 2, -1]^T$.

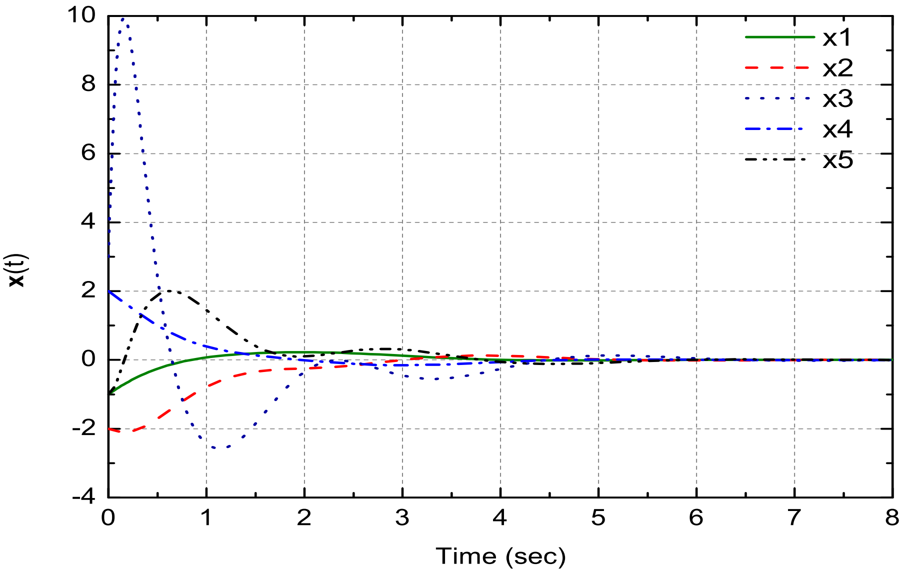

FIGURE 4.9: Closed loop response of the extended nonholonomic double integrator with gain $k = 2$ and initial condition: $x = [-1, -2, 3, 2, -1]^T$.

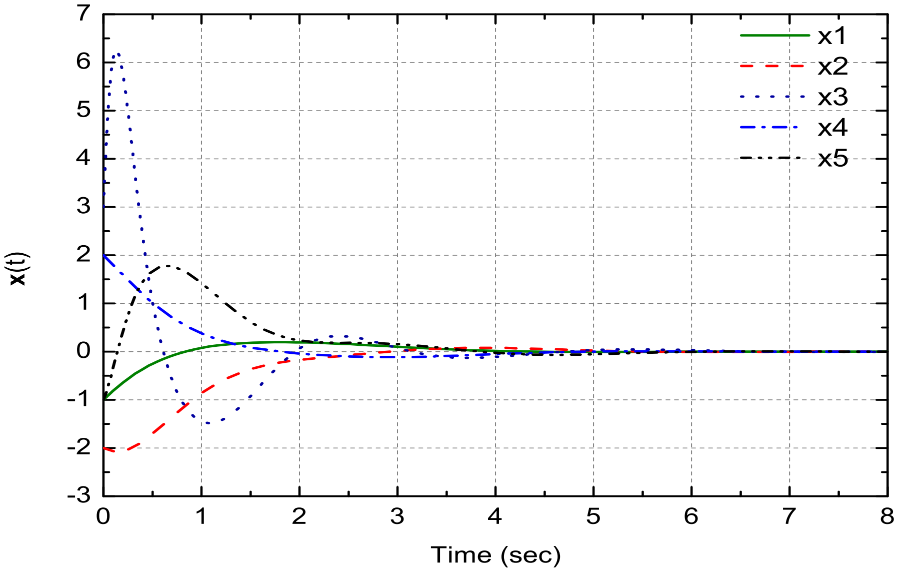

FIGURE 4.10: Closed loop response of the extended nonholonomic double integrator with gain $k = -2$ and initial condition:
$x = [-1, -2, 3, 2, -1]^T$.

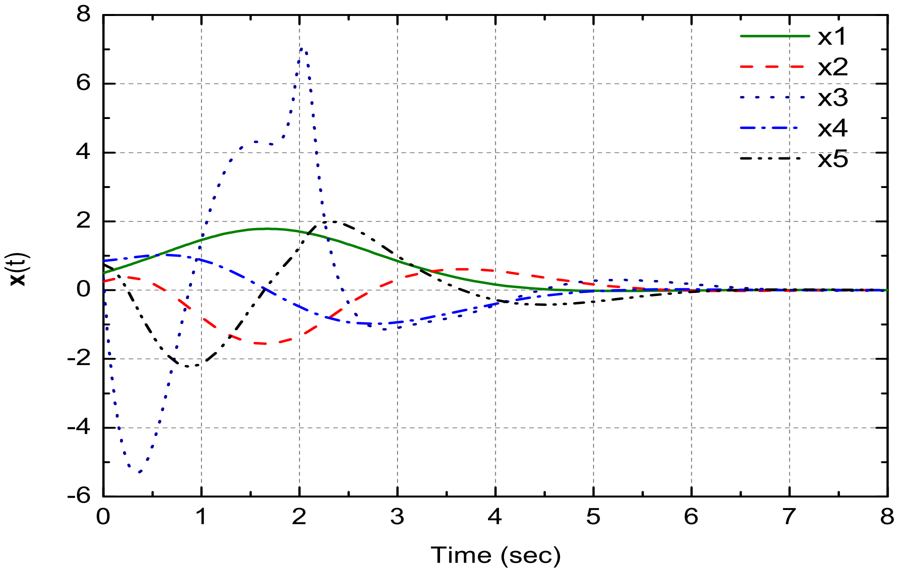

FIGURE 4.11: Closed loop response of the extended nonholonomic double integrator with gain $k = -\hat{\theta}$ and initial condition:
$x = [.5, .2, .3, .9, .8]^T$.

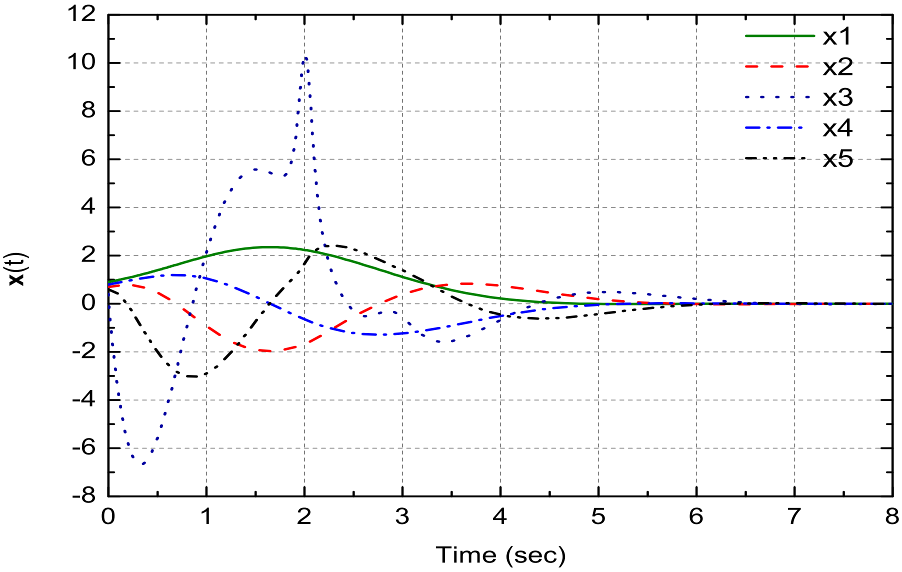

FIGURE 4.12: Closed loop response of the extended nonholonomic double integrator for initial condition: $x = [0.9, 0.7, 0.4, 0.8, 0.6]^T$.

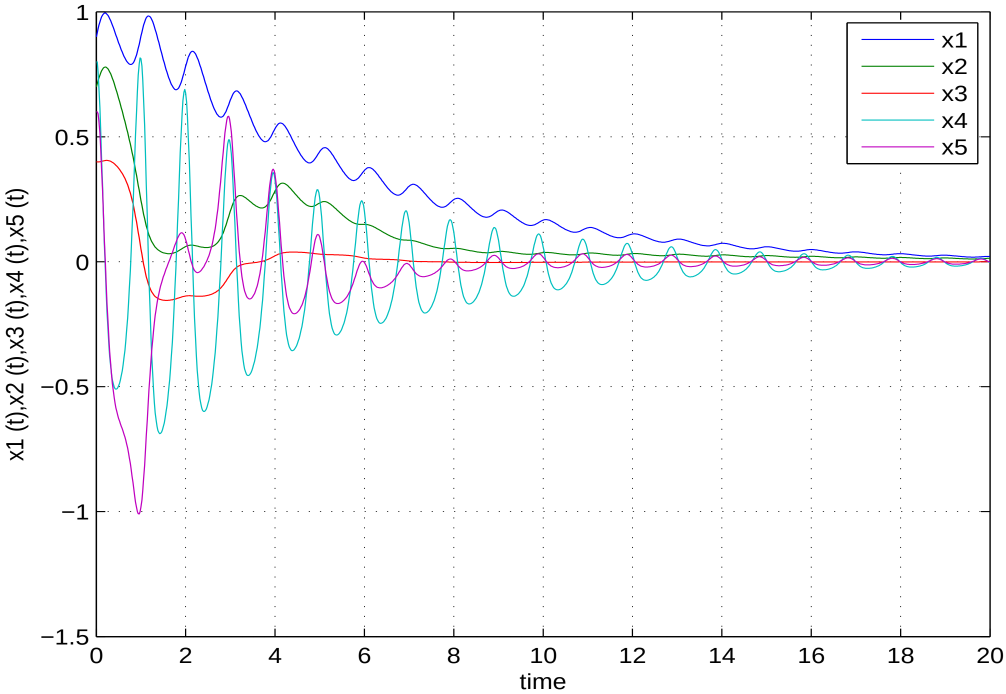

FIGURE 4.13: The simulation results for continuous controller [115] for initial conditions $x = [0.9, 0.7, 0.4, 0.8, 0.6]^T$.

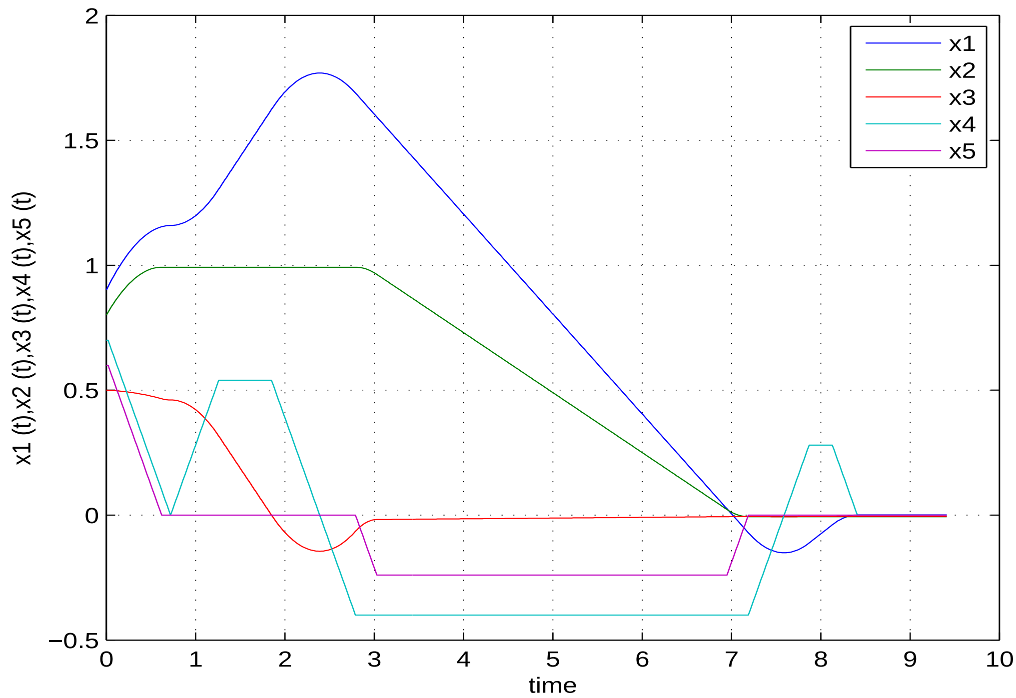

FIGURE 4.14: The simulation results for discontinuous controller [115] for initial conditions $x = [0.9, 0.7, 0.4, 0.8, 0.6]^T$.

4.6 Summary

In this chapter, a control algorithm for the stabilization of systems with drift is presented and is applied to a rigid body and an extended nonholonomic double integrator model. A comparison analysis has also been done with the existing method to show the effectiveness of the proposed algorithm. By using an adaptive backstepping technique, a time-varying transformation is constructed to transform an original system into a new system which is asymptotically stabilized at the origin. Simulation results show the validness of the proposed method.

Chapter 5

Stabilization of Nonholonomic Systems: Adaptive Integral Sliding Mode Control Technique

5.1 Introduction

This chapter presents an adaptive integral sliding mode control algorithm for the stabilization of nonholonomic drift free systems. First, the system is transformed, by using input transforms, into a particular structure containing a nominal part and some unknown terms, which are computed adaptively. The transformed system is then stabilized using adaptive integral sliding mode control. On the basis of Lyapunov stability, the compensator control and the adaptive laws are derived. The control algorithm is applied to three drift-free systems, namely; a unicycle model, a front-wheel car model, and a mobile robot with trailer model. The effectiveness of proposed algorithm is verified through numerical simulations.

5.2 Problem Formulation

5.2.1 Mathematical Model of Nonholonomic Systems

The kinematic model for a drift-free nonholonomic system is given as:

$$\dot{x} = \sum_{i=1}^{m} g_i(x)\, u_i, \qquad x \in \Re^n \tag{5.1}$$

Where, $g_i(x)$ represents a vector field on $\Re^n$, u_i are a continuous control input from interval $[0, \infty)$. The main hurdle in the stabilization of these systems is that linearization of system (5.1) is uncontrollable. The most difficult issue from a theoretical viewpoint is the design of feedback laws that can stabilize these systems about an equilibrium position.

5.2.2 Problem Statement

For a desired point $x_{des} \in \Re^n$, a control input $v_i : \Re^n \to \Re$, $i = 1, 2, ..., m$ is constructed in such a way that x_{des} is an attractive point for (5.1), as $t \to \infty$ leads to $x(t;\ 0,\ x_0) \to x_{des}$. Furthermore, by suitable transformation of the system, $x_{des} = 0$ can be achieved.

5.2.3 Assumptions

The system (5.1) must fulfill the following conditions.

- (P1): The vector fields $g_1(x)$, ..., $g_m(x)$ must be linearly independent.
- (P2): System (5.1) must satisfy the Lie algebra rank condition (LARC): $L(g_1, ..., g_m)(x)$ spans $\Re^n$ at each point $x \in \Re^n$.

5.3 The Proposed Control Algorithm

Step 1:

Write the system (5.1) as:

$$\begin{aligned} \dot{x}_1 &= g_1(x,u) \\ \dot{x}_2 &= g_2(x,u) \\ &\vdots \\ \dot{x}_{n-1} &= g_{n-1}(x,u) \\ \dot{x}_n &= g_n(x,u) \end{aligned} \tag{5.2}$$

where, $g_i : \Re^n \times \Re^m \to R$ are nonlinear functions.

Step 2:

Using the input transformation, the system (5.2) can be written the following form:

$$\begin{aligned} \dot{x}_1 &= h_1(x) \\ \dot{x}_2 &= h_2(x) \\ &\vdots \\ \dot{x}_{n-1} &= h_{n-1}(x) \\ \dot{x}_n &= v \end{aligned} \tag{5.3}$$

where $h_i : \Re^n \times \Re^m \to R$ are nonlinear functions and v is the new input. After some manipulation, system (5.3) can be rewritten as:

$$\begin{aligned} \dot{x}_1 &= x_2 + F_1 \\ \dot{x}_2 &= x_3 + F_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n + F_{n-1} \\ \dot{x}_n &= v \end{aligned} \tag{5.4}$$

Where $F_i = -x_{i+1} + h_i(x)$.

Step 3:

Assume that F_i are uncertainties in the system. Let $\hat{F}_i$, $i = 1, ..., n$ be an estimate of F_i, $i = 1, ..., n-1$ respectively. Apply the function approximation technique [116] to represent F_i and their estimates $\hat{F}_i$ as: $F_i = w_i^T \varphi_i(t)$ and $\hat{F}_i = \hat{w}_i^T \varphi_i(t)$. Where, $\varphi_i(t) = \begin{bmatrix} \varphi_{i1}(t) & \varphi_{i2}(t) & \cdots & \varphi_{in}(t) \end{bmatrix}^T$ is a basis vector and $w_i = \begin{bmatrix} w_{i1} & w_{i2} & \cdots & w_{in} \end{bmatrix}^T$ is a weight vector. Let $\hat{w}_i = \begin{bmatrix} \hat{w}_{i1} & \hat{w}_{i2} & \cdots & \hat{w}_{in} \end{bmatrix}^T$ be estimate of $w_i = \begin{bmatrix} w_{i1} & w_{i2} & \cdots & w_{in} \end{bmatrix}^T$. Therefore we can estimate F_i by estimating the weight vector w_i i.e. $\hat{F}_i = \hat{w}_i^T \varphi_i(t)$. Define $\tilde{w}_i = w_i - \hat{w}_i$, then system (5.4) is written as:

$$\begin{aligned} \dot{x}_1 &= x_2 + \hat{w}_1^T \varphi_1(t) + \tilde{w}_1^T \varphi_1(t) \\ \dot{x}_2 &= x_3 + \hat{w}_2^T \varphi_2(t) + \tilde{w}_2^T \varphi_2(t) \\ &\vdots \\ \dot{x}_{n-1} &= x_n + \hat{w}_{n-1}^T \varphi_{n-1}(t) + \tilde{w}_{n-1}^T \varphi_{n-1}(t) \\ \dot{x}_n &= v \end{aligned} \tag{5.5}$$

Step 4:

Choose the nominal system for (5.5) as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= v_0 \end{aligned} \tag{5.6}$$

Step 5:

Define the sliding manifold for the nominal system (5.6) as:

$$s_0 = x_1 + \sum_{i=2}^{n-1} c_i x_i + x_n \tag{5.7}$$

where $c_i > 0$. Then

$$\dot{s}_0 = \dot{x}_1 + \sum_{i=2}^{n-1} c_i\dot{x}_i + \dot{x}_n = x_2 + \sum_{i=2}^{n-1} c_i x_{i+1} + v_0 \tag{5.8}$$

By choosing

$$v_0 = -x_2 - \sum_{i=2}^{n-1} c_i x_{i+1} - k\, sign(s_0), \quad k > 0 \tag{5.9}$$

We have

$$\dot{s}_0 = -k\, sign(s_0) \tag{5.10}$$

Therefore Eq. (5.6) shows that the nominal system is asymptotic stable.

Step 6:

Define the sliding manifold for system (5.5) as:

$$s = s_0 + z = x_1 + \sum_{i=2}^{n-1} c_i x_i + x_n + z \tag{5.11}$$

Where z is an integral term. To avoid the reaching phase, choose $z(0)$ such that $s(0) = 0$. Choose $v = v_0 + v_s$ where, v_0 is the nominal input and v_s is compensator term. Then

$$\begin{aligned}
\dot{s} &= \dot{x}_1 + \sum_{i=2}^{n-1} c_i\dot{x}_i + \dot{x}_n + \dot{z} \\
&= x_2 + \hat{w}_i^T\varphi_1(t) + \tilde{w}_i^T\varphi_2(t) + \sum_{i=2}^{n-1} c_i\{x_{i+1} + \hat{w}_i^T\varphi_i(t) + \tilde{w}_i^T\varphi_i(t)\} + v_0 + v_s + \dot{z} \\
&= x_2 + \sum_{i=2}^{n-1} c_i x_{i+1} + v_0 + v_s + \dot{z} + \sum_{i=1}^{n-1}\{\hat{w}_i^T\varphi_i(t) + \tilde{w}_i^T\varphi_i(t)\}
\end{aligned} \tag{5.12}$$

where $c_1 = 1$.

Step 7:

Choose a Lyapunov function as:

$$V = \frac{1}{2}s^2 + \frac{1}{2}\sum_{i=1}^{n-1} \tilde{w}_i^T\tilde{w}_i \tag{5.13}$$

Design the adaptive laws for $\tilde{w}_i$ & $\hat{w}_i,\ i = 1, ..., n$ and compute the value of v_s such that $\dot{V} < 0$.

Theorem 5.1. *Choose a Lyapunov function as:*

$$V = \frac{1}{2}s^2 + \frac{1}{2}\sum_{i=1}^{n-1} \tilde{w}_i^T \tilde{w}_i \tag{5.14}$$

Then $\dot{V} < 0$ if the adaptive laws for $\tilde{w}_i$ & $\hat{w}_i$, and the value of v_s are chosen as:

$$\begin{aligned}
\dot{z} &= -x_2 - \sum_{i=2}^{n-1} c_i x_{i+1} - v_0 \\
v_s &= -\sum_{i=1}^{n-1} c_i \hat{w}_i^T \varphi_i(t) - k\, sign(s) \\
\dot{\tilde{w}}_i &= -c_i s \varphi_i(t) - k_i\, \tilde{w}_i \\
\dot{\hat{w}}_i &= -\dot{\tilde{w}}_i
\end{aligned} \tag{5.15}$$

where k and $k_i > 0, \quad i = 1, ..., n-1.$

Proof: Since

$$\begin{aligned}
\dot{V} &= s\dot{s} + \sum_{i=1}^{n-1} \tilde{w}_i^T \dot{\tilde{w}}_i = s\{x_2 + \hat{w}_1^T \varphi_1(t) + \tilde{w}_1^T \varphi_1(t) \\
&+ \sum_{i=2}^{n-1} c_i\{x_{i+1} + \hat{w}_i^T \varphi_i(t) + \tilde{w}_i^T \varphi_i(t)\} + v_0 + v_s + \dot{z}\} + \sum_{i=1}^{n-1} \tilde{w}_i^T \dot{\tilde{w}}_i \\
&= s\{x_2 + \sum_{i=1}^{n-1} \{c_i x_{i+1} + c_i \hat{w}_i^T \varphi_i(t)\} + v_0 + v_s + \dot{z}\} \\
&+ \sum_{i=1}^{n-1} \tilde{w}_i^T \{\dot{\tilde{w}}_i + c_i s \varphi_i(t)\}
\end{aligned} \tag{5.16}$$

By using

$$\begin{aligned}
\dot{z} &= -x_2 - \sum_{i=2}^{n-1} c_i x_{i+1} - v_0 \\
v_s &= -\sum_{i=1}^{n-1} c_i \hat{w}_i^T \varphi_i(t) - k\, sign(s) \\
\dot{\tilde{w}}_i &= -c_i s \varphi_i(t) - k_i\, \tilde{w}_i \\
\dot{\hat{w}}_i &= -\dot{\tilde{w}}_i
\end{aligned} \tag{5.17}$$

where k and $k_i > 0, \quad i = 1, ..., n-1$. we have

$$\dot{V} = -k\,|s| - \sum_{i=1}^{n-1} k_i \tilde{w}_i^T \tilde{w}_i \tag{5.18}$$

Choosing:

$$k_n = \min(k, k_1, ..., k_{n-1})$$

We have:

$$\dot{V} \leq -k_n(\,|s| + \sum_{i=1}^{n-1} \tilde{w}_i^T \tilde{w}_i) \tag{5.19}$$

From this we conclude that s & $\tilde{w}_i \to 0$, $i = 1, ..., n$. Since $s \to 0$, therefore $x \to 0$.

In the following section, we illustrate the above algorithm by applying it to three different nonholonomic drift free systems.

5.4 Application to Nonholonomic Systems

5.4.1 Unicycle Model

A unicycle model or a two-wheel car model, shown in Fig. 5.1, is basically a three-dimensional nonholonomic system having two inputs and three states with depth one Lie bracket. A two-wheel car kinematic model is defined as [117].

$$\begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix} u_2 \tag{5.20}$$

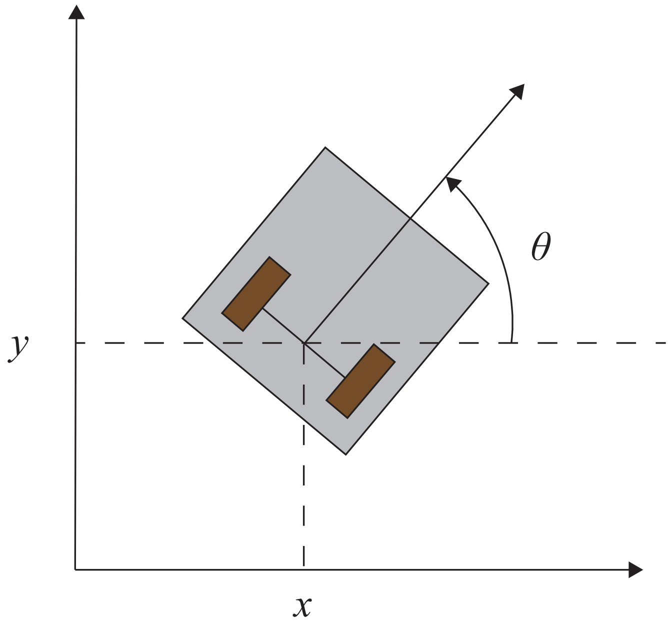

FIGURE 5.1: The schematics of unicycle model

By choosing state variables as: $x \stackrel{def}{=} [x_1, x_2, x_3]^T = [\theta,\ x,\ y]^T$, the kinematics model (5.20) can be written as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \cos x_1 \\ \sin x_1 \end{bmatrix} u_2 \tag{5.21}$$

Above equation can be represented as:

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2, \quad x \in \Re^3 \tag{5.22}$$

where, $g_1(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $g_2(x) = \begin{bmatrix} 0 \\ \cos x_1 \\ \sin x_1 \end{bmatrix}$.

The kinematics model (5.20) satisfies the assumptions presented in section 5.2.3.

To verify both properties (P1) and (P2), calculate the Lie bracket as: $g_1(x)$ and $g_2(x)$.

$$g_3(x) \overset{def}{=} [g_1, g_2](x) = \begin{bmatrix} 0 \\ -\sin x_1 \\ \cos x_1 \end{bmatrix}$$

Then the LARC namely: $span\{g_1, g_2, g_3\}(x) = \Re^3, \forall \;\; x \in \Re^3$ is satisfied.

5.4.1.1 Application of the Proposed Algorithm to a Unicycle Model

Step 1:

The unicycle model (5.21) can be rewritten as:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= \cos x_1 u_2 \\ \dot{x}_3 &= \sin x_1 u_2 \end{aligned} \tag{5.23}$$

Step 2:

Choose $u_1 = v$ and $u_2 = \frac{x_3}{\cos x_1}$, where, $x_1 \neq \frac{\pi}{2}$, then system (5.23) becomes:

$$\begin{aligned} \dot{x}_1 &= v \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_3 \tan x_1 \end{aligned} \tag{5.24}$$

After some manipulation the above mentioned system can be written as:

$$\begin{aligned} \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1 + F \\ \dot{x}_1 &= v \end{aligned} \tag{5.25}$$

where $F = -x_1 + x_3 \tan x_1$.

Step 3:

Assuming F as an uncertainty and let $\hat{F}$ be an estimate of F. The estimate of F is computed by function approximating technique [116] as: $F = w^T\varphi$ and $\hat{F} = \hat{w}^T\varphi$, then the system (5.25) can be written as:

$$\begin{aligned} \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1 + \hat{w}^T\varphi + \tilde{w}^T\varphi \\ \dot{x}_1 &= v \end{aligned} \tag{5.26}$$

Step 4:

Choose the nominal system for (5.26) as:

$$\begin{aligned} \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1 \\ \dot{x}_1 &= v_0 \end{aligned} \tag{5.27}$$

Step 5:

Define the sliding manifold for the nominal system (5.27) as:

$$s_0 = x_2 + 2x_3 + x_1 \tag{5.28}$$

Then:

$$\dot{s}_0 = \dot{x}_2 + 2\dot{x}_3 + \dot{x}_1 = x_3 + 2x_1 + v_0 \tag{5.29}$$

By choosing:

$$v_0 = -x_3 - 2x_1 - k\, sign(s_0), \quad k > 0 \tag{5.30}$$

We have $\dot{s}_0 = -k\, sign(s_0)$. Therefore (5.27) shows that the nominal system is asymptotically stable.

Step 6:

Define the sliding manifold for system (5.26) as:

$$s = s_0 + z = x_2 + 2x_3 + x_1 + z \tag{5.31}$$

Choose:

$$v = v_0 + v_s$$

Then

$$\dot{s} = \dot{x}_1 + 2\dot{x}_3 + \dot{x}_2 + \dot{z} = x_3 + 2x_2 + 2\hat{w}^T\varphi + 2\tilde{w}^T\varphi + v_0 + v_s + \dot{z} \tag{5.32}$$

Step 7:

The following adaptive laws for $\tilde{w}, \hat{w}$ and the value of v_s

$$\begin{aligned} \dot{z} &= -x_3 - 2x_2 - v_0 \\ v_s &= -2\hat{w}^T\varphi - k\, sign(s) \\ \dot{\tilde{w}} &= -2s\varphi - k_1\,\tilde{w} \\ \dot{\hat{w}} &= -\dot{\tilde{w}} \end{aligned} \tag{5.33}$$

where k and $k_1 > 0$

Gives

$$\dot{V} = -\,k\,|s| - k_1\tilde{w}^T\tilde{w} \tag{5.34}$$

Where

$$V = \frac{1}{2}s^2 + \frac{1}{2}\tilde{w}^T\tilde{w} \tag{5.35}$$

Choosing:

$$k_2 = \min(k, k_1)$$

we have:

$$\dot{V} \leq -\,k_2(\,|s| + \tilde{w}^T\tilde{w}) \tag{5.36}$$

From the above equation it is concluded that s & $\tilde{w} \to 0$. Since $s \to 0$, therefore $x \to 0$. Numerical results are shown in Fig. 5.4 and Fig. 5.5.

5.4.2 Front Wheel Car Model

A front-wheel car model, shown in Fig. 5.2 is basically a four-dimensional non-holonomic system having two inputs and four states with depth two Lie bracket. A front-wheel car kinematic model [113] can be defined as:

$$\begin{bmatrix} \dot{\psi} \\ \dot{x} \\ \dot{\theta} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \cos\theta \\ \frac{1}{l}\tan\psi \\ \sin\theta \end{bmatrix} u_2 \tag{5.37}$$

Assuming that $l = 1$ and choose a new state variables as: $x \overset{def}{=} (x_1, x_2, x_3, x_4) =$

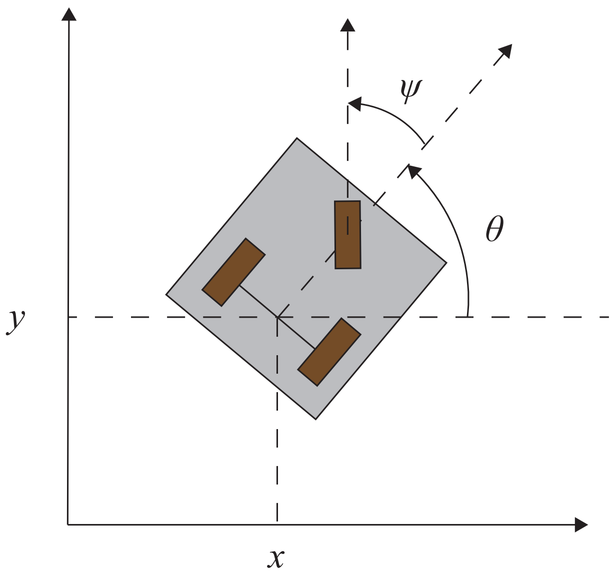

FIGURE 5.2: The schematics of front wheel car model

$(\psi,\, x,\, y, \theta)$ the kinematics model (5.37) is written as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \cos x_4 \\ \sin x_4 \\ \tan x_1 \end{bmatrix} u_2 \tag{5.38}$$

or we can write the above Eq. (5.38) as:

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2, \qquad x \in \Re^4 \tag{5.39}$$

where, $g_1(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $g_2(x) = \begin{bmatrix} 0 \\ \cos x_4 \\ \sin x_4 \\ \tan x_1 \end{bmatrix}$

The kinematics model (5.38) satisfies the assumptions presented in section 5.2.3. To verify both properties (P1) and (P2), the Lie brackets $g_1(x)$ & $g_2(x)$ are calculated as follows:

$$g_3(x) \overset{def}{=} [g_1, g_2](x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ (\sec x_1)^2 \end{bmatrix}, \quad g_4(x) \overset{def}{=} [g_2, g_3](x) = \begin{bmatrix} 0 \\ -\sin x_4 (\sec x_1)^2 \\ \cos x_4 (\sec x_1)^2 \\ 0 \end{bmatrix}$$

From above, the LARC is satisfied: $span(g_1, g_2, g_3, g_4)(x) = \Re^4 \ \forall \ x \in \Re^4$.

5.4.2.1 Application of the Proposed Algorithm to a Front Wheel Car Model

Step 1:

The front wheel car model as given in (5.38) is represented as:

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= \cos x_4 u_2 \\
\dot{x}_3 &= \sin x_4 u_2 \\
\dot{x}_4 &= \tan x_1 u_2
\end{aligned} \tag{5.40}$$

Step 2:

Choose $u_1 = v$ and $u_2 = \frac{x_3}{\cos x_4}$, where $x_4 \neq \frac{\pi}{2}$, then system (5.40) becomes:

$$\begin{aligned}
\dot{x}_1 &= v \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_3 \tan x_4 \\
\dot{x}_4 &= x_3 \tan x_1 \sec x_4
\end{aligned} \tag{5.41}$$

which can be rewritten as:

$$\begin{aligned}
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 + F_3 \\
\dot{x}_4 &= x_1 + F_4 \\
\dot{x}_1 &= v
\end{aligned} \tag{5.42}$$

where

$$\begin{aligned}
F_3 &= -x_4 + x_3 \tan x_4 \\
F_4 &= -x_1 + x_3 \tan x_1 \sec x_4
\end{aligned} \tag{5.43}$$

Step 3:

Treating $F_i,\ i = 3,4$ as uncertainties and let $\hat{F}_i,\ i = 3,4$ be an estimate of $F_i,\ i = 3,4$ respectively. Using function approximation technique [116], we can approximate $F_i,\ i = 3,4$ as: $F_3 = w_3^T \varphi_3,\ F_4 = w_4^T \varphi_4$. Then $\hat{F}_3 = \hat{w}_3^T \varphi_3$ and $\hat{F}_4 = \hat{w}_4^T \varphi_4$.

Then the system (5.42) can be written as:

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_3 &= x_4 + \hat{w}_3^T \varphi_3 + \tilde{w}_3^T \varphi_3 \\ \dot{x}_4 &= x_2 + \hat{w}_4^T \varphi_4 + \tilde{w}_4^T \varphi_4 \\ \dot{x}_2 &= v \end{aligned} \tag{5.44}$$

Step 4:

Choose the nominal system for (5.44) as:

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= x_2 \\ \dot{x}_2 &= v_0 \end{aligned} \tag{5.45}$$

Step 5:

Define the sliding manifold for the nominal system (5.45) as:

$$s_0 = x_1 + 3x_3 + 3x_4 + x_2 \tag{5.46}$$

Then

$$\dot{s}_0 = \dot{x}_1 + 3\dot{x}_3 + 3\dot{x}_4 + \dot{x}_2 = x_3 + 3x_4 + 3x_2 + v_0 \tag{5.47}$$

By choosing

$$v_0 = -x_3 - 3x_4 - 3x_2 - k\, sign(s_0), \quad k > 0 \tag{5.48}$$

we have

$$\dot{s}_0 = -k\, sign(s_0). \tag{5.49}$$

Therefore Eq. (5.45) shows that the nominal system is asymptotically stable.

Step 6:

Define the sliding manifold for system (5.44) as:

$$s = s_0 + z = x_1 + 3x_3 + 3x_4 + x_2 + z \tag{5.50}$$

Choose:

$$v = v_0 + v_s$$

Then

$$\begin{aligned}\dot{s} = \dot{x}_1 + 3\dot{x}_3 + 3\dot{x}_4 + \dot{x}_2 + \dot{z} = x_3 + 3x_4 + 3\hat{w}_3^T\varphi_3 + 3\tilde{w}_3^T\varphi_3 \\ +3x_2 + 3\hat{w}_4^T\varphi_4 + 3\tilde{w}_4^T\varphi_4 + v_0 + v_s + \dot{z}\end{aligned} \tag{5.51}$$

Step 7:

The following adaptive laws for $\tilde{w}_i, \hat{w}_i,\ \ i = 3, 4$ and the value of v_s are chosen as:

$$\begin{aligned}\dot{z}\ &= -x_3 - 3x_4 - 3x_2 - v_0 \\ v_s &= -3\hat{w}_3^T\varphi_3 - 3\hat{w}_4^T\varphi_4 - k\, sign(s) \\ \dot{\tilde{w}}_3 &= -3s\varphi_3 - k_1\,\tilde{w}_3^T \\ \dot{\hat{w}}_3 &= -\dot{\tilde{w}}_3 \\ \dot{\tilde{w}}_4 &= -3s\varphi_4 - k_2\tilde{w}_4^T \\ \dot{\hat{w}}_4 &= -\dot{\tilde{w}}_4\end{aligned} \tag{5.52}$$

where k, k_1 and $k_2 > 0$. Gives

$$\dot{V} = -\,k\,|s| - k_1\tilde{w}_3^T\tilde{w}_3 - k_2\tilde{w}_4^T\tilde{w}_4 \tag{5.53}$$

where

$$V = \frac{1}{2}s^2 + \frac{1}{2}\tilde{w}_3^T\tilde{w}_3 + \frac{1}{2}\tilde{w}_4^T\tilde{w}_4 \tag{5.54}$$

Choosing: $k_3 = \min(k, k_1, k_2)$, we have:

$$\dot{V} \le -\,k_3(\,|s| + k_1\tilde{w}_3^T\tilde{w}_3 + k_2\tilde{w}_4^T\tilde{w}_4) \tag{5.55}$$

From this, we conclude that $s, \tilde{w}_3, \tilde{w}_4 \to 0$. Since $s \to 0$, therefore $x \to 0$. Results are shown in Fig. 5.6 and Fig. 5.7.

5.4.3 Mobile Robot with Trailer Model

A car with trailer model, shown in Fig. 5.3, is basically a five-dimensional non-holonomic system having two inputs and five states with depth one, two and three Lie brackets. A car with trailer kinematic model [118] can be defined as:

$$\begin{aligned} \dot{x}_1 &= \cos x_3 \cos x_4 \; u_1 \\ \dot{x}_2 &= \cos x_3 \sin x_4 \; u_1 \\ \dot{x}_3 &= \; u_2 \\ \dot{x}_4 &= \tfrac{1}{l} \sin x_3 \; u_1 \\ \dot{x}_5 &= \tfrac{1}{d} \sin(x_4 - x_5) \cos x_3 \; u_1 \end{aligned} \tag{5.56}$$

By assuming $l = d = 1$, system (5.56) is represented as:

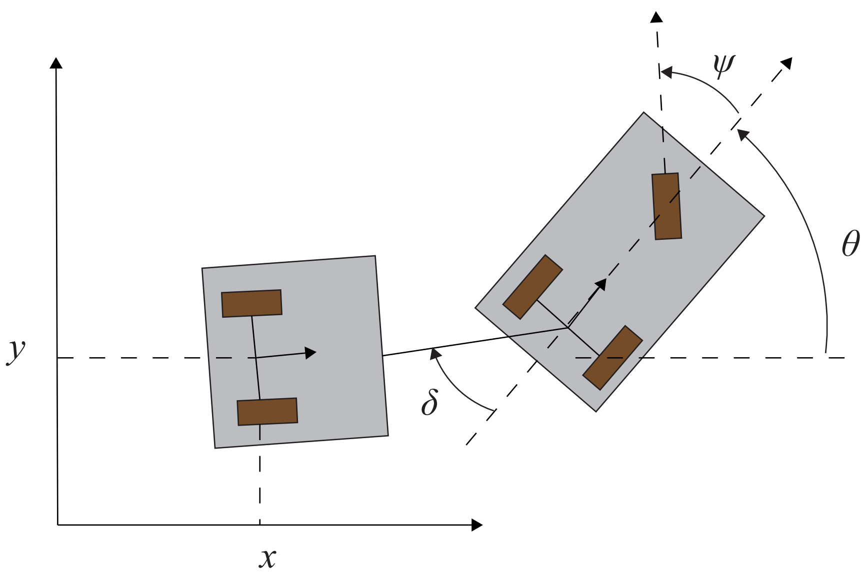

FIGURE 5.3: The schematics of mobile robot with trailer Model

$$\dot{x} = g_1(x)\, u_1 + g_2(x)\, u_2\,, \qquad x \in \Re^5 \tag{5.57}$$

where,

$$g_1(x) = \begin{bmatrix} \cos x_3 \cos x_4 \\ \cos x_3 \sin x_4 \\ 0 \\ \sin x_3 \\ \cos x_3 \sin(x_4 - x_5) \end{bmatrix} \text{ and } g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The kinematics model (5.57) satisfies the assumptions presented in section 5.2.3. To verify both properties (P1) and (P2), calculate the following Lie brackets as:

$$g_3(x) \stackrel{\text{def}}{=} [g_1, g_2](x) = \begin{bmatrix} \sin x_3 \cos x_4 \\ \sin x_3 \sin x_4 \\ 0 \\ -\cos x_3 \\ \sin x_3 \sin(x_4 - x_5) \end{bmatrix}$$

$$g_4(x) \stackrel{\text{def}}{=} [g_1, [g_1, g_2]](x) = \begin{bmatrix} -\sin x_4 \\ \cos x_4 \\ 0 \\ 0 \\ \cos(x_4 - x_5) \end{bmatrix}$$

$$g_5(x) \stackrel{\text{def}}{=} [g_1, [g_1, [g_1, g_2]]](x) = \begin{bmatrix} -\sin x_3 \cos x_4 \\ -\sin x_3 \sin x_4 \\ 0 \\ 0 \\ -\sin x_3 \sin(x_4 - x_5) + \cos x_3 \end{bmatrix}$$

If movement of the system is restricted to surface: $\mathrm{M} \stackrel{def}{=} \{x \in \Re^5 : |x_i| < \frac{\pi}{2}, i = 3,\ 4\}$, then the Lie algebra rank condition namely $span\,\{g_1(x), g_2(x), \ldots.., g_5(x)\} = \Re^5, \quad \forall \quad x \in M$ is satisfied, hence guarantees that system (5.57) satisfy the conditions P1 and P2 on the surface M.

5.4.3.1 Application of the Proposed Algorithm to a Mobile Robot with Trailer Model

Step 1:

The system (5.56) can be written as:

$$\begin{aligned}
\dot{x}_1 &= \cos x_3 \cos x_4 \; u_1 \\
\dot{x}_2 &= \cos x_3 \sin x_4 \; u_1 \\
\dot{x}_3 &= u_2 \\
\dot{x}_4 &= \sin x_3 \; u_1 \\
\dot{x}_5 &= \sin(x_4 - x_5) \cos x_3 \; u_1
\end{aligned} \tag{5.58}$$

Step 2:

Choose $u_1 = \frac{x_2}{\cos x_3 \cos x_4}$ and $u_2 = v$. Where, $x_3, x_4 \neq \frac{\pi}{2}$. Then system (5.58) can be written as:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_2 \tan x_4 \\
\dot{x}_3 &= v \\
\dot{x}_4 &= x_2 \tan x_3 \sec x_4 \\
\dot{x}_5 &= x_2 \sin(x_4 - x_5) \sec x_4
\end{aligned} \tag{5.59}$$

which can be rewritten as:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_4 + F_1 \\
\dot{x}_4 &= x_5 + F_2 \\
\dot{x}_5 &= x_3 + F_3 \\
\dot{x}_3 &= v
\end{aligned} \tag{5.60}$$

Step 3:

Assuming F_i, $i = 2, 4, 5$ as uncertainties and let $\hat{F}_i$, $i = 2, 4, 5$ be an estimate of F_i, $i = 2, 4, 5$ respectively. Approximating $F_i = w_i^T \varphi_i$, $i = 2, 4, 5$ and let

$\hat{F}_i = \hat{w}_i^T \varphi_i,\ i = 2, 4, 5$ respectively. Then the system (5.60) can be written as:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_4 + \hat{w}_2^T \varphi_2 + \tilde{w}_2^T \varphi_2\\ \dot{x}_4 &= x_5 + \hat{w}_4^T \varphi_4 + \tilde{w}_4^T \varphi_4\\ \dot{x}_5 &= x_3 + \hat{w}_5^T \varphi_5 + \tilde{w}_5^T \varphi_5\\ \dot{x}_3 &= \ v\end{aligned} \tag{5.61}$$

Step 4:

Choose the nominal system for (5.61) as:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_4\\ \dot{x}_4 &= x_5\\ \dot{x}_5 &= x_3\\ \dot{x}_3 &= \ v_0\end{aligned} \tag{5.62}$$

Step 5:

Define the Hurwitz sliding manifold for the nominal system (5.62) as:

$$s_0 = x_1 + 4x_2 + 6x_4 + 4x_5 + x_3 \tag{5.63}$$

Then

$$\begin{aligned}\dot{s}_0 &= \dot{x}_1 + 4\dot{x}_2 + 6\dot{x}_4 + 4\dot{x}_5 + \dot{x}_3\\ &= x_2 + 4x_4 + 6x_5 + 4x_3 + v_0\end{aligned} \tag{5.64}$$

By choosing

$$v_0 = -x_2 - 4x_4 - 6x_5 - 4x_3 - k\, sign(s_0), \quad k > 0 \tag{5.65}$$

we have

$$\dot{s}_0 = -k\, sign(s_0) \tag{5.66}$$

Therefore the nominal system (5.62) is asymptotically stable.

Step 6:

Define the sliding surface for the system (5.61) as:

$$s = s_0 + z = x_1 + 4x_2 + 6x_4 + 4x_5 + x_3 + z \tag{5.67}$$

and choose

$$v = v_0 + v_s$$

Then

$$\begin{aligned} \dot{s} &= \dot{x}_1 + 4\dot{x}_2 + 6\dot{x}_4 + 4\dot{x}_5 + \dot{x}_3 + \dot{z} \\ &= x_2 + 4x_4 + 4\hat{w}_2^T\varphi_2 + 4\tilde{w}_2^T\varphi_2 + 6x_5 + 6\hat{w}_4^T\varphi_4 + 6\tilde{w}_4^T\varphi_4 \\ &+4x_3 + 4\hat{w}_5^T\varphi_5 + 4\tilde{w}_5^T\varphi_5 + v_0 + v_s + \dot{z} \end{aligned} \tag{5.68}$$

Step 7: The following adaptive laws for $\tilde{w}_i \& \hat{w}_i$, and the value of v_s.

$$\begin{aligned} &\dot{z} = -x_2 - 4x_4 - 6x_5 - 4x_3 - v_0 \\ &v_s = -4\hat{w}_2^T\varphi_2 - 6\hat{w}_4^T\varphi_4 - 4\hat{w}_5^T\varphi_5 - k\, sign(s) \\ &\dot{\tilde{w}}_2 = -4s\varphi_2 - k_1\, \tilde{w}_2^T \\ &\dot{\hat{w}}_2 \approx -\dot{\tilde{w}}_2 \\ &\dot{\tilde{w}}_4 = -6s\varphi_4 - k_2\, \tilde{w}_4^T \\ &\dot{\hat{w}}_4 \approx -\dot{\tilde{w}}_4 \\ &\dot{\tilde{w}}_5 = -4s\varphi_5 - k_3\, \tilde{w}_5^T \\ &\dot{\hat{w}}_5 \approx -\dot{\tilde{w}}_5 \end{aligned} \tag{5.69}$$

where k and $k_i > 0, \quad i = 1, 2, 3,$ Gives

$$\dot{V} = -\,k\,|s| - k_1\tilde{w}_2^T\tilde{w}_2 - k_2\tilde{w}_4^T\tilde{w}_4 - k_3\tilde{w}_5^T\tilde{w}_5 \tag{5.70}$$

where

$$V = \frac{1}{2}s^2 + \frac{1}{2}\tilde{w}_2^T\tilde{w}_2 + \frac{1}{2}\tilde{w}_4^T\tilde{w}_4 + \frac{1}{2}\tilde{w}_5^T\tilde{w}_5 \tag{5.71}$$

Choosing: $k_4 = \min(k, k_1, k_2, k_3)$, we have:

$$\dot{V} \le -\,k_4(\;|s| + \tilde{w}_2^T\tilde{w}_2 + \tilde{w}_4^T\tilde{w}_4 + \tilde{w}_5^T\tilde{w}_5) \tag{5.72}$$

From this we conclude that $s, \tilde{w}_2^T, \tilde{w}_4^T$ and $\tilde{w}_5^T \to 0$. Since $s \to 0$, therefore $x \to 0$. Simulation results are shown in Fig. 5.8 and Fig. 5.9.

5.5 Simulation Results

Fig. 5.4 and Fig. 5.5 show results of the unicycle model and represent that the states, and the control effort converge to zero and have been settling time of 4 sec and 0.8 sec and have gains $k = 5$ and $k_1 = 2$. The basis vector is chosen as $\varphi = cos(2\pi t)$. Fig. 5.6 and Fig. 5.7 show results of the front-wheel car model and represent that the states, and the control effort converge to zero and have been settling time of 6 sec and 1 sec and have gains $k = 5$, $k_1 = 3$ and $k_2 = 3$. The basis vectors are chosen as $\varphi_3 = cos(2\pi t)$ and $\varphi_4 = sin(2\pi t)$. Fig. 5.8 and Fig. 5.9 show results for the car with trailer model and represent that the states and control effort converge to zero and have been settling time of 10 sec and 0.4 sec and has gains $k = 5$, $k_1 = 1$, $k_2 = 4$ and $k_3 = 6$. The basis vectors are chosen as $\varphi_2 = cos(2\pi t)$, $\varphi_4 = sin(2\pi t)$ and $\varphi_4 = sin(4\pi t)$. Numerical results show the effectiveness of the proposed scheme.

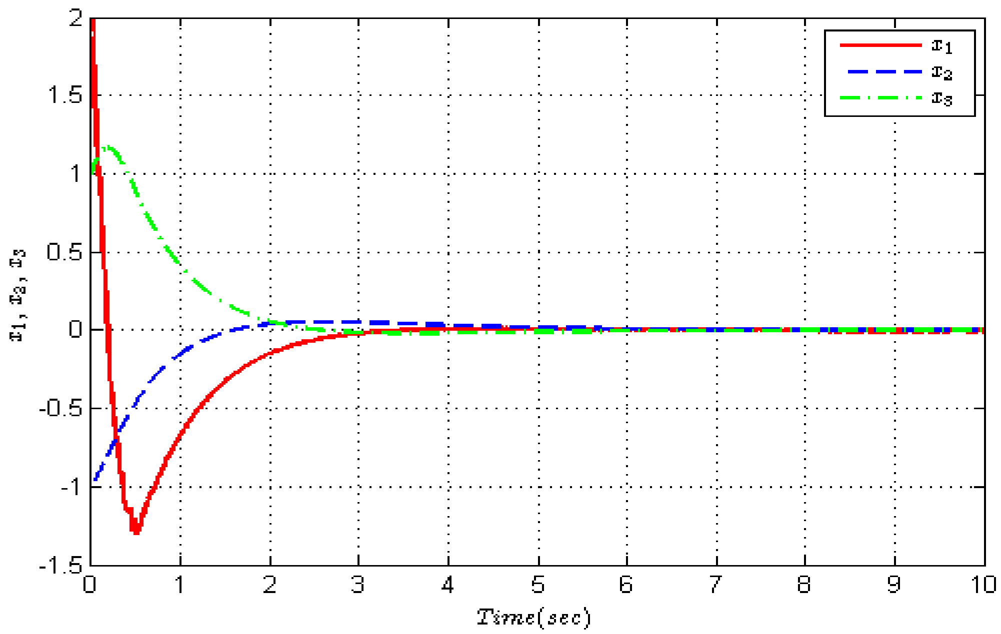

FIGURE 5.4: Closed loop response of the unicycle model for initial condition: $(x_1(0), x_2(0), x_3(0)) = (2, -1, 1)$

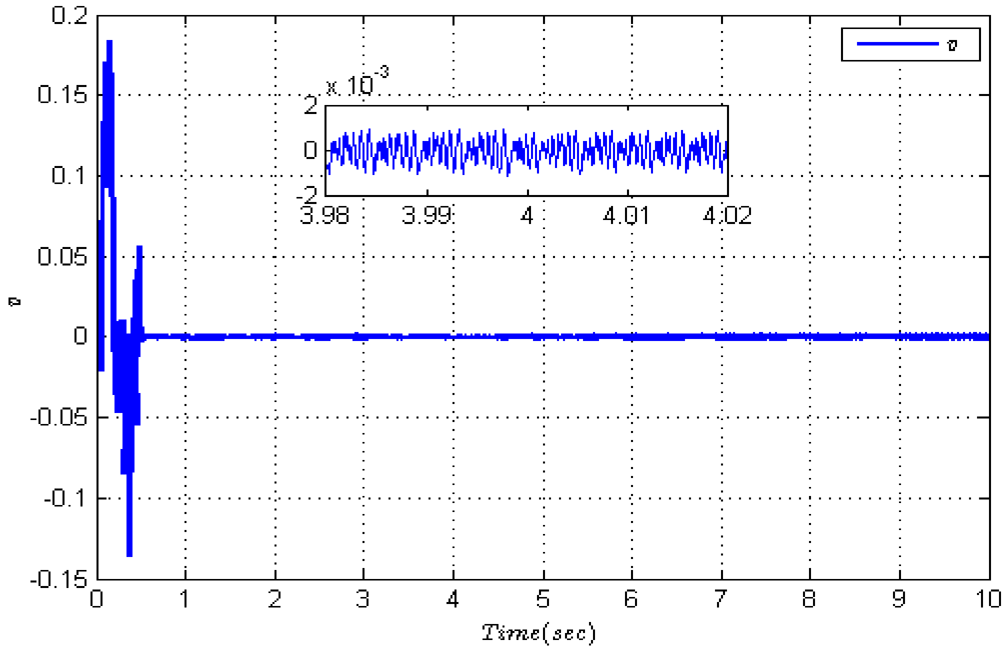

FIGURE 5.5: Control input v

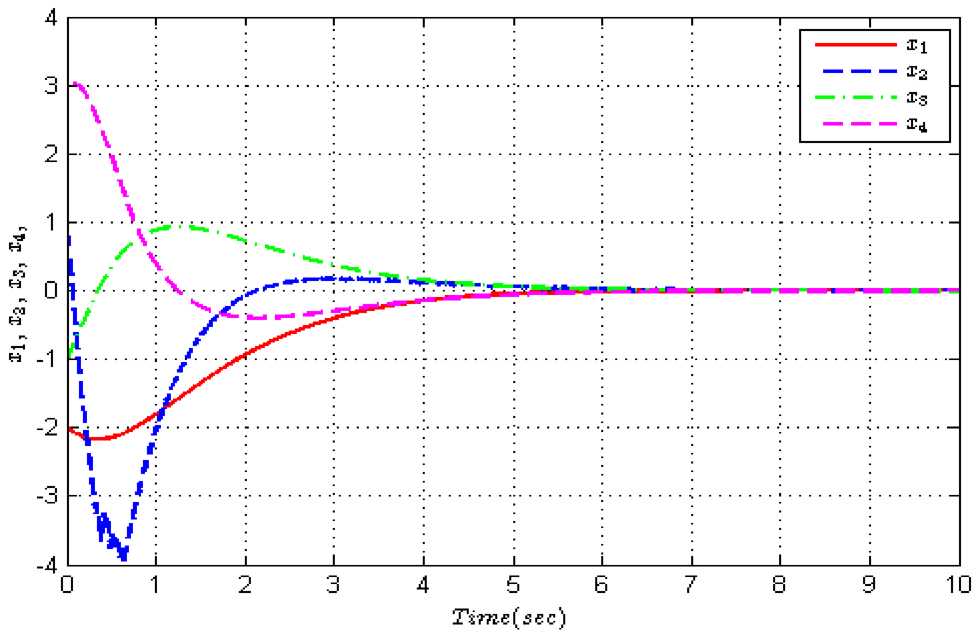

FIGURE 5.6: Closed loop response of the front wheel car model for initial condition: $(x_1(0), ..., x_4(0)) = (-2, 1, -1, 3)$

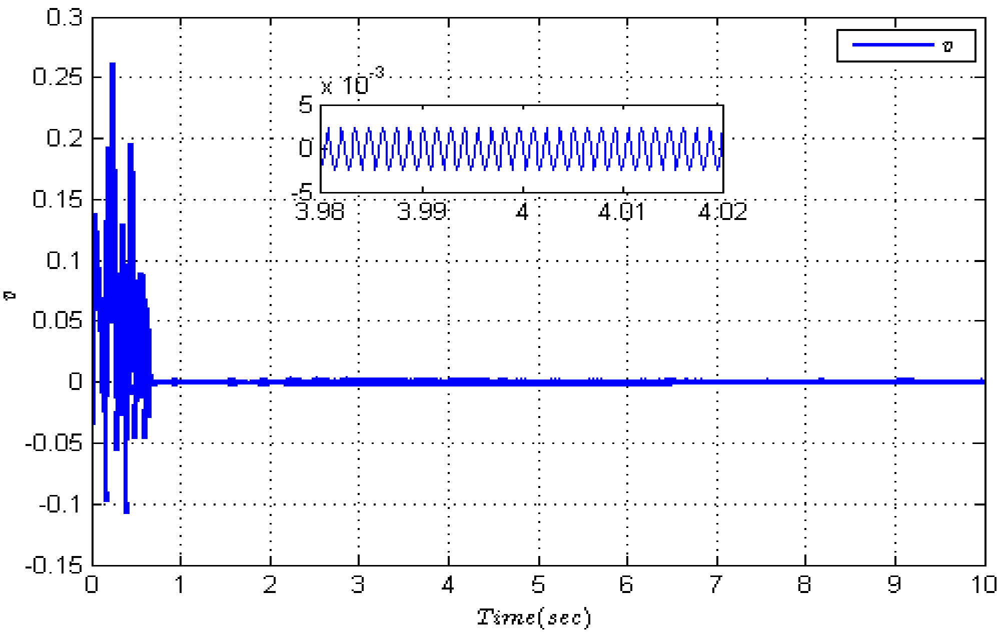

FIGURE 5.7: Control input v

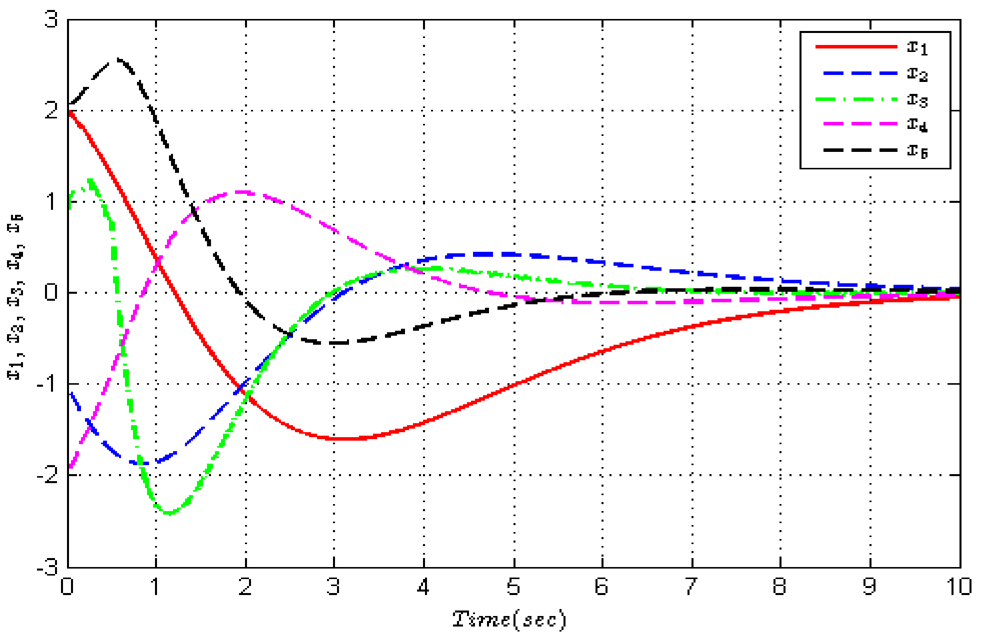

FIGURE 5.8: Closed loop response of the car with trailer model for initial condition: $(x_1(0), ..., x_5(0)) = (2, -1, 1, -2, 2)$

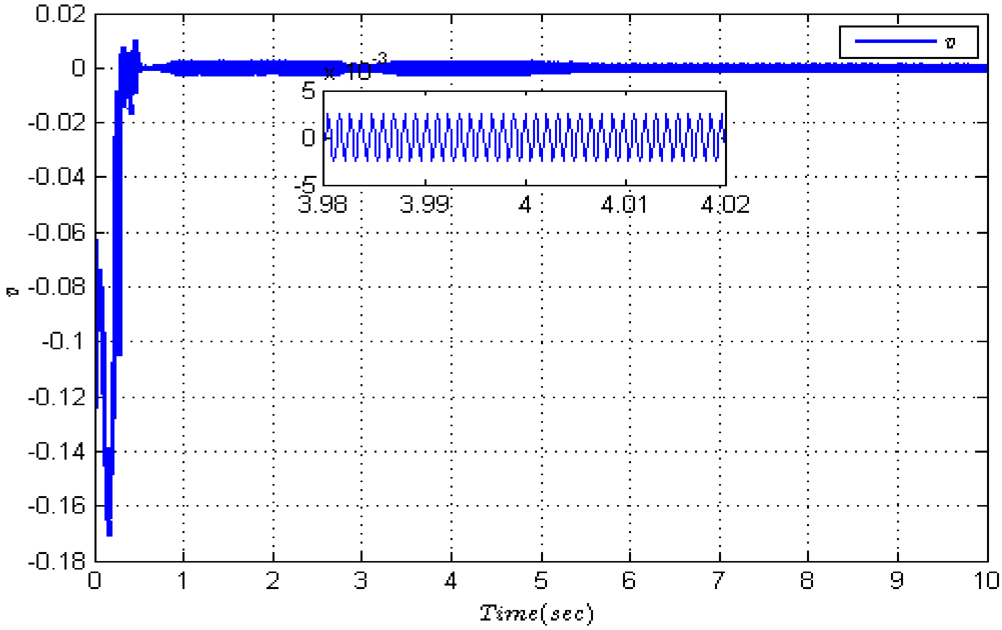

FIGURE 5.9: Control input v

5.6 Summary

An adaptive integral sliding mode based approach for the stabilization of drift-free nonholonomic control systems is presented in this chapter. The objective is to stabilize the nonholonomic systems to the origin which is assumed to be zero from any initial state. The proposed algorithm is applied to three examples of drift-free nonholonomic systems; the unicycle, the front-wheel car, and the mobile robot with trailer. It is shown from simulation results that the objective has been achieved. This strategy is general and can be applied to stabilize a variety of nonholonomic systems.

Chapter 6

Stabilization of Nonholonomic Systems: Smooth Super Twisting Sliding Mode Control Technique

6.1 Introduction

In the present chapter, a new solution to the stabilizing control problem of nonholonomic systems, which are transformable into chained form is investigated. A smooth super twisting sliding mode control technique is used to stabilize nonholonomic systems. Firstly, the nonholonomic system is transformed into a chained form system, which is further decomposed into two subsystems. Secondly, the second subsystem is stabilized to the origin by using smooth super twisting sliding mode control. Finally, the first subsystem is stabilized to zero using signum function. The stated algorithm is tested on three different nonholonomic systems, which are transformable into chained form; a two-wheel car model, a model of front-wheel car, and a firetruck model. Numerical computer simulations show the effectiveness of the proposed control scheme.

6.2 The Control Problem Formulation

The kinematics model of nonholonomic systems in presence of constraints is represented as:

$$\dot{z} = \sum_{i=1}^{m} g_i(z)\, u_i, \qquad z \in \Re^n \tag{6.1}$$

where $g_i,\ i = 1, \cdots, m$ represents vector fields on $\Re^n$, u_i are the continuous control input from interval $[0,\ \infty)$. In control system design it is a very useful technique that the system is first transformed into some canonical form via state-input transformation. One such canonical form is chained form introduced first time by [113].

$$\begin{aligned} \dot{x}_1 &= v_1 \\ \dot{x}_2 &= v_2 \\ \dot{x}_3 &= x_2 v_1 \\ \dot{x}_4 &= x_3 v_1 \\ &\vdots \\ \dot{x}_n &= x_{n-1} v_1 \end{aligned} \tag{6.2}$$

For a desired point $x_{des} \in \Re^n$, a control input $v_i : \Re^n \to \Re,\ i = 1, 2$ is constructed in such a way that x_{des} is an attractive point for (6.2), as $t \to \infty$ lead to $x(t;\ 0,\ x_0) \to x_{des}$. Furthermore, by suitable transformation of the system, $x_{des} = 0$ can be achieved.

Assumptions:

The system (6.1) must fulfill the following conditions.

- (P1): The state vector fields $g_1(z)$,..., $g_m(z)$ must be linearly independent.
- (P2): System (6.1) must satisfy the Lie algebra rank condition (LARC): $L(g_1, ..., g_m)(z)$ spans $\Re^n$ at each point of $z \in \Re^n$.

6.3 The Proposed Stabilizing Control Algorithm

Step 1:

Choose $v_1 = 1$ and $v_2 = v$, then system (6.2) becomes:

$$\begin{aligned}
\dot{x}_1 &= 1 \\
\dot{x}_2 &= v \\
\dot{x}_3 &= x_2 \\
\dot{x}_4 &= x_3 \\
&\vdots \\
\dot{x}_{n-1} &= x_{n-1} \\
\dot{x}_n &= x_{n-1}
\end{aligned} \tag{6.3}$$

Step 2:

Decompose the system (6.3) into two subsystems as:

$$\begin{aligned}
&S_1 : \dot{x}_1 = 1 \\
&S_2 : \begin{cases} \dot{x}_n = x_{n-1} \\ \dot{x}_{n-1} = x_{n-2} \\ \vdots \\ \dot{x}_4 = x_3 \\ \dot{x}_3 = x_2 \\ \dot{x}_2 = v \end{cases}
\end{aligned} \tag{6.4}$$

Define the sliding surface for subsystem S_2 as:

$$s = x_n + \sum_{i=1}^{n-3} c_i x_{n-i} + x_2 \tag{6.5}$$

where $c_i > 0$ are chosen so that s becomes Hurwitz polynomial. Then

$$\dot{s} = \dot{x}_n + \sum_{i=1}^{n-3} c_i \dot{x}_{n-i} + \dot{x}_2 = x_{n-1} + \sum_{i=1}^{n-3} c_i x_{n-1-i} + v \tag{6.6}$$

By choosing

$$v = v_{eq} + v_s \tag{6.7}$$

where

$$v_{eq} = -x_{n-1} - \sum_{i=1}^{n-3} c_i x_{n-1-i} \tag{6.8}$$

$$v_s = -k_1 |s|^{\frac{\rho-1}{\rho}} \, sign(s) \, + z \tag{6.9}$$

$$\dot{z} = -k_2 \, |s|^{\frac{\rho-2}{\rho}} sign(s) \tag{6.10}$$

where $k_1, k_2 > 0$, $\rho \geq 2$.

we have

$$\dot{s} = -k_1 |s|^y \, sign(s) \, + z \tag{6.11}$$

$$\dot{z} = -k_2 \, |s|^{y-\frac{1}{\rho}} sign(s) \tag{6.12}$$

where $k_1, k_2 > 0$, $y = \frac{\rho-1}{\rho}$. Therefore the subsystem S_2 is stable.

The stability proof of system (6.11) and (6.12) is based on the context given in reference [119].

Step 3:

Apply the control inputs:

$$v_1 = 1 \tag{6.13}$$

$$v = v_{eq} + v_s \tag{6.14}$$

where

$$v_{eq} = -x_{n-1} - \sum_{i=1}^{n-3} c_i x_{n-1-i} \tag{6.15}$$

$$v_s = -k_1 |s|^{\frac{\rho-1}{\rho}} sign(s) + z \tag{6.16}$$

$$\dot{z} = -k_2 \, |s|^{\frac{\rho-2}{\rho}} sign(s) \tag{6.17}$$

where $k_1, k_2 > 0,\ \rho \geq 2$, until the system (6.3) reaches on a surface:

$$S = \{x \in \Re^n : x_n = ... = x_3 = x_2 = 0\, , x_1 \neq 0\}$$

Step 4:

Apply the control inputs $v_1 = -ksign(x_1)$ and $v = 0$ until the system (6.3) reaches at origin:

$$O = \{x \in \Re^n : x_n = ... = x_3 = x_2 = x_1 = 0\}$$

6.4 Application Examples

6.4.1 A Two Wheel Car Model

6.4.1.1 A Kinematics Model

A unicycle model or a two-wheel car model shown in Fig. 6.1 represents a three dimensional nonholonomic system. The kinematic model of a two-wheel car is given as [41]:

$$\begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix} u_2 \tag{6.18}$$

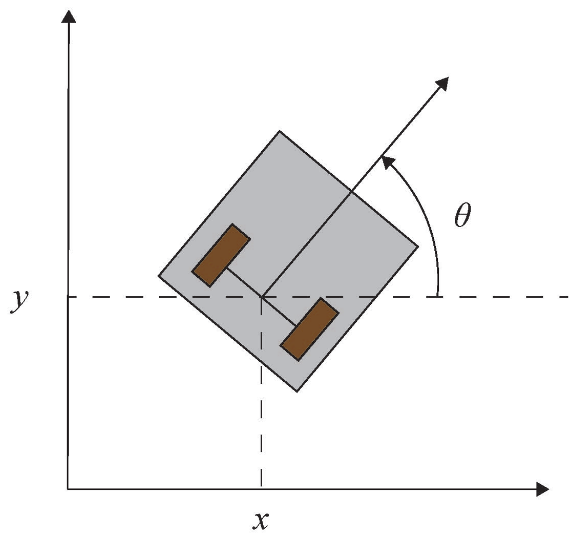

FIGURE 6.1: The schematics of two wheel car model

Introducing a new set of state variables $z = [z_1, z_2, z_3]^T = [\theta,\ x,\ y]^T$ the kinematics model (6.18) can be written as:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \cos z_1 \\ \sin z_1 \end{bmatrix} u_2 \tag{6.19}$$

or

$$\dot{z} = g_1(z)u_1 + g_2(z)u_2,\ z \in \Re^3$$

where,

$$\mathrm{g_c(z)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathrm{g_2(z)} = \begin{bmatrix} 0 \\ \cos z_1 \\ \sin z_1 \end{bmatrix}$$

6.4.1.2 Conversion into Chained form

Consider the following transformation:

$$x_1 = z_1 \tag{6.20}$$

$$x_2 = z_2 \cos z_1 + z_3 \sin z_1 \tag{6.21}$$

$$x_3 = z_2 \sin z_1 - z_3 \cos z_1 \tag{6.22}$$

$$v_1 = u_1 \tag{6.23}$$

$$v_2 = u_2 - z_3 u_1 \tag{6.24}$$

This transforms the nonholonomic system (6.20) into the following chained form system:

$$\dot{x}_1 = v_1 \tag{6.25}$$

$$\dot{x}_2 = v_2 \tag{6.26}$$

$$\dot{x}_3 = x_2 v_1 \tag{6.27}$$

Then, the proposed algorithm can be used with the following sliding surface:

$$s = x_3 + x_2 \tag{6.28}$$

6.4.1.3 Simulation Results

Simulation results of proposed algorithm for different initial conditions are shown in Fig. 6.2 and Fig. 6.4. In Fig. 6.2 and Fig. 6.4 the initial conditions are chosen as $[x_1(0), x_2(0), x3(0)] = [1, 2, -4]$ and $[x_1(0), x_2(0), x3(0)] = [2, -4, 3]$. Fig. 6.3 and Fig. 6.5 shows the control inputs v_1, v_2 with gains $k_1 = 1$, $k_2 = 5$. From the figures, we can see that under the constructed controller, the solution process of the closed-loop system converges to zero almost surely.

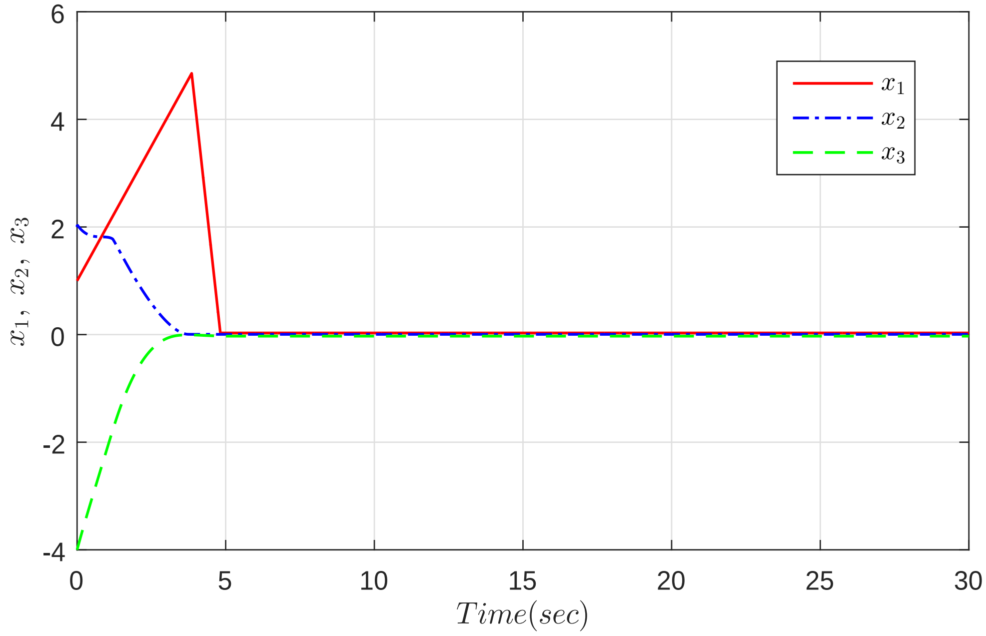

FIGURE 6.2: Closed loop response of the two wheel car model for initial condition: $(x_1, x_2, x_3) = (1, 2 - 4)$

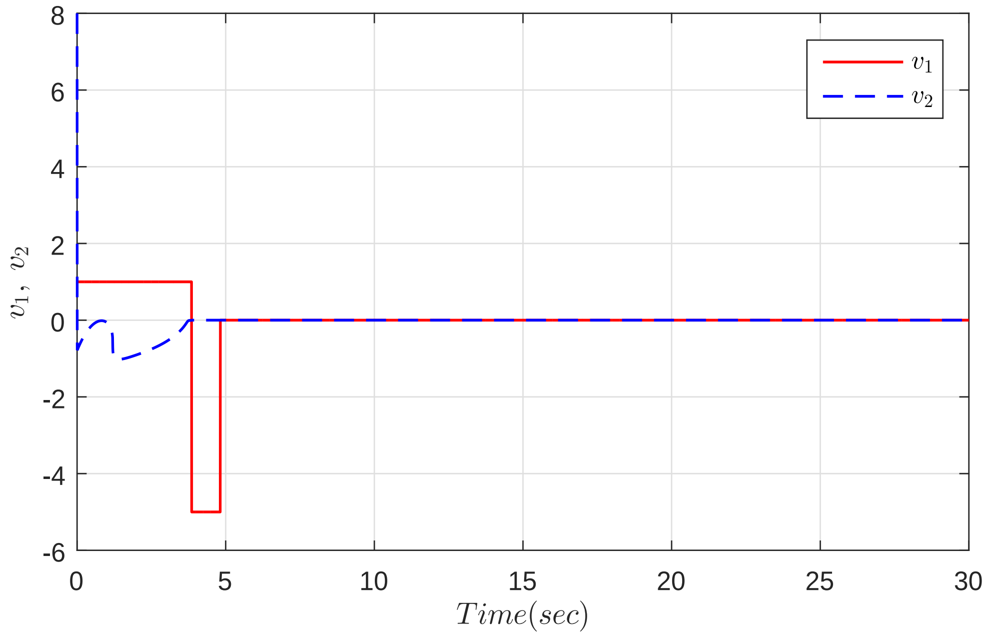

FIGURE 6.3: Control effort v_1, v_2

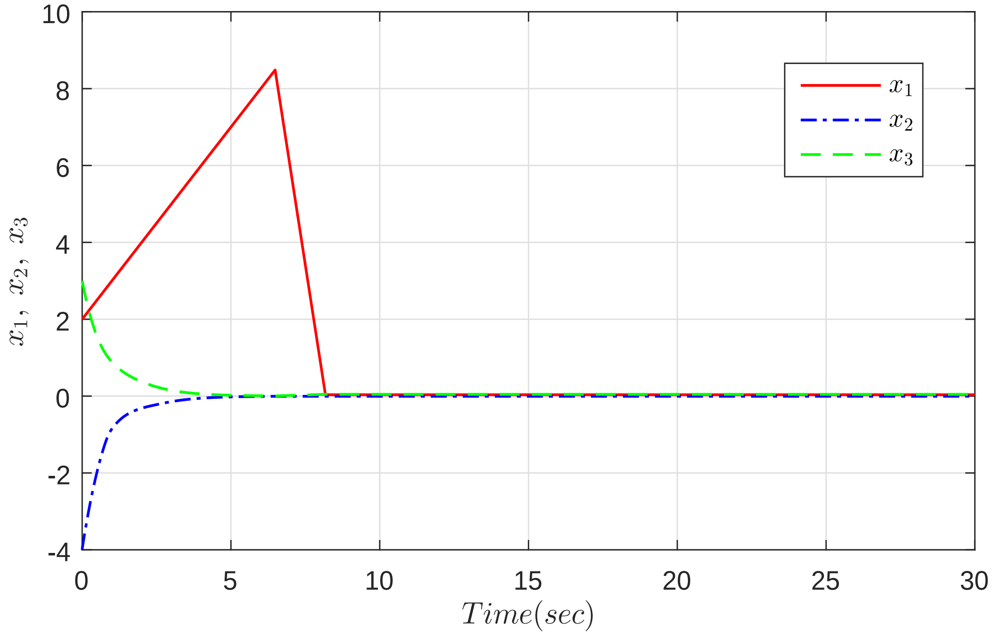

FIGURE 6.4: Closed loop response of the two wheel car model for initial condition: $(x_1, x_2, x_3) = (2, -4, 3)$

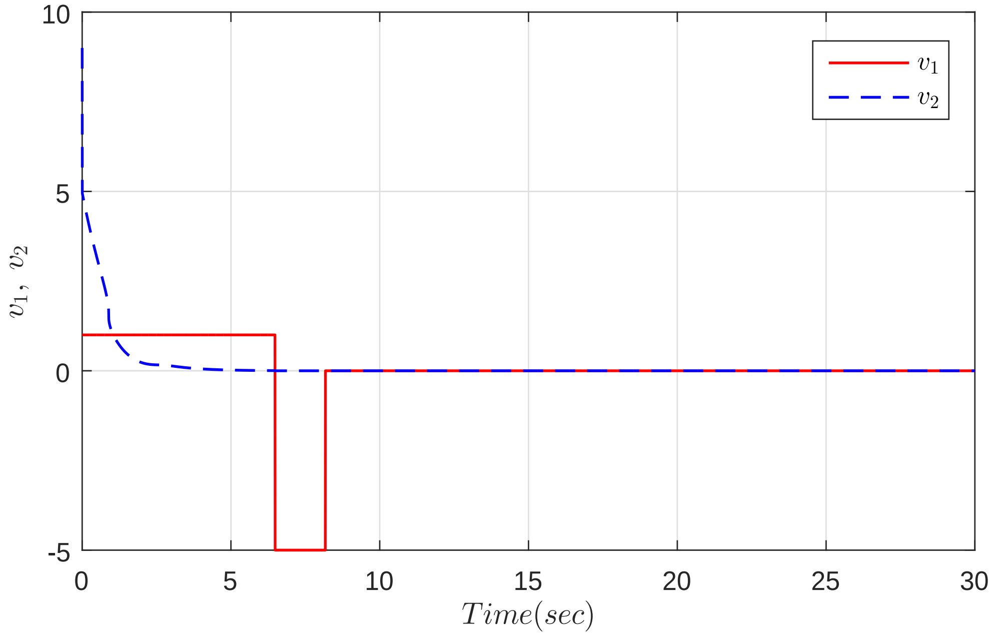

FIGURE 6.5: Control effort v_1, v_2

6.4.2 A Front Wheel Car Model

6.4.2.1 A Kinematics Model

The front wheel car model as shown in Fig. 6.6 represents four-dimensional non-holonomic system. The kinematics model of a front wheel car model is given as in [120]:

$$\begin{bmatrix} \dot{\varphi} \\ \dot{x} \\ \dot{\theta} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \cos\theta \\ \frac{1}{l}\tan\varphi \\ \sin\theta \end{bmatrix} u_2 \tag{6.29}$$

where,

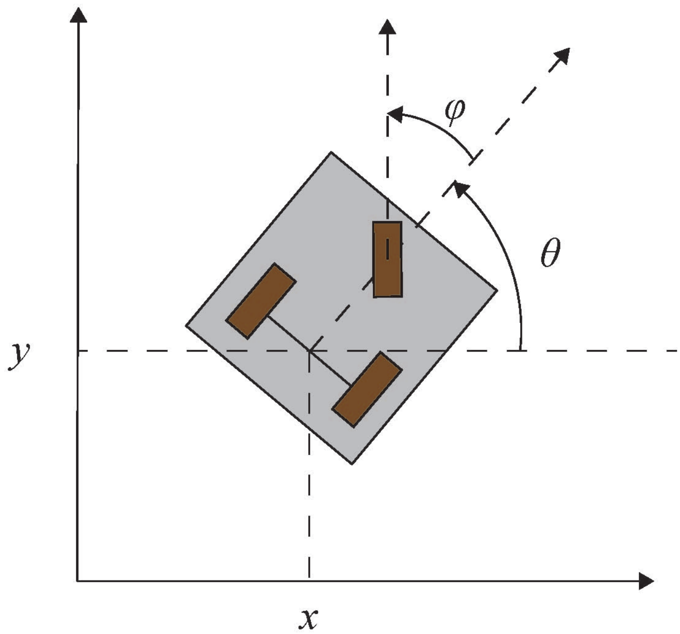

FIGURE 6.6: The schematics of front wheel car model

(x, y): the center coordinate of the rear axle of the car

φ: the steering angle of the front wheel of the car

θ: the orientation of the car body with respect to x-axis

l: the distance between the front and rear axles of the car.

Assuming that $l = 1$and introducing a new set of state variables $z \overset{def}{=} (z_1, z_2, z_3,\ z_4) = (\varphi,\ x,\ y, \theta)$ the kinematics model (6.29) can be written as:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \cos z_4 \\ \sin z_4 \\ \tan z_1 \end{bmatrix} u_2 \tag{6.30}$$

or

$$\dot{z} = g_1(z)u_1 + g_2(z)u_2, z \in \Re^4$$

where, $\mathrm{g}_1(z) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ & $\mathrm{g}_2(z) = \begin{bmatrix} 0 \\ \cos z_4 \\ \sin z_4 \\ \tan z_1 \end{bmatrix}$

6.4.2.2 Conversion into Chained form

Consider the following transformation:

$$\begin{aligned} x_1 &= x = z_2 \\ x_2 &= \tfrac{\tan\varphi}{l\cos^3\theta} = \tfrac{\tan z_1}{l\cos^3 z_4} \\ x_3 &= \tan\theta\ = \tan z_4 \\ x_4 &= y = x_3 \\ v_1 &= \cos\theta\, u_1 = \cos z_4\, u_1 \\ v_2 &= \tfrac{1+\tan^2\varphi}{l\cos^3\theta} u_1 + \tfrac{3\tan\theta\tan^2\varphi}{l^2\cos^3\theta} u_2 \\ &= \tfrac{1+\tan^2 z_1}{l\cos^3 z_4} u_1 + \tfrac{3\tan z_4\tan^2 z_1}{l^2\cos^3 z_4} u_2 \end{aligned} \tag{6.31}$$

Transforms the nonholonomic system (6.31) into the following chained form system:

$$\begin{aligned} \dot{x}_1 &= v_1 \\ \dot{x}_2 &= v_2 \\ \dot{x}_3 &= x_2 v_1 \\ \dot{x}_4 &= x_3 v_1 \end{aligned} \tag{6.32}$$

The proposed algorithm was applied to sliding surface:

$$s = x_4 + 2x_3 + x_2 \tag{6.33}$$

6.4.2.3 Simulation Results

Fig. 6.7 and Fig. 6.9 shows the response of system states when proposed algorithm is applied. The initial conditions are chosen as $[x_1(0), x_2(0), x_3(0), x_4(0)] = [0.3, 0.4, 0.2, 0.5]$ and $[x_1(0), x_2(0), x_3(0), x_4(0)] = [1, -2, 3, -4]$. Fig. 6.8 and Fig. 6.10 shows the control inputs v_1, v_2 with gains $k_1 = 4$, $k_2 = 10$. From the figures, we can see that under the constructed controller, the solution process of the closed-loop system converges to zero almost surely.

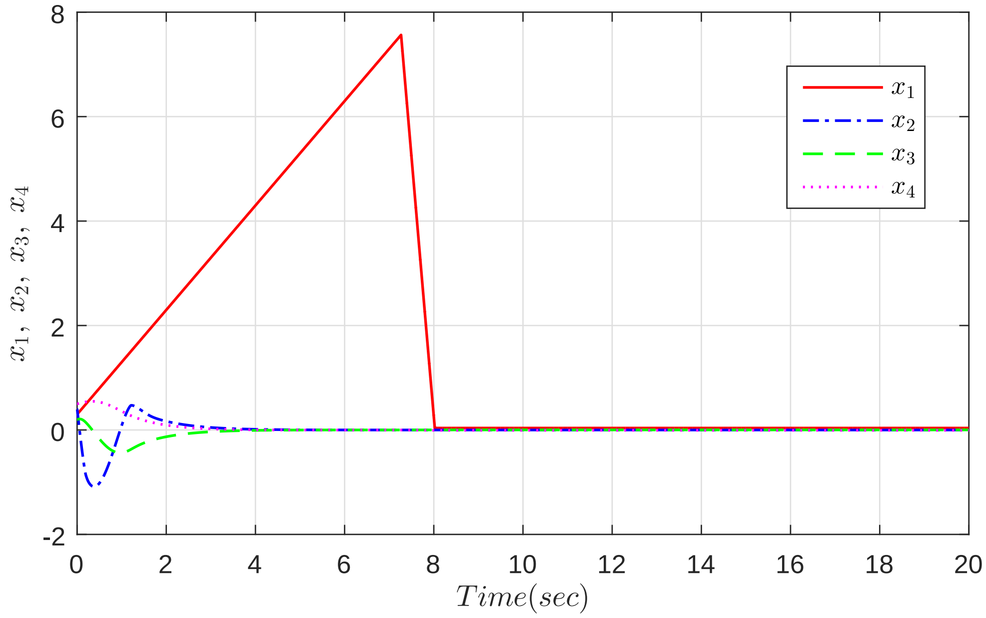

FIGURE 6.7: Closed loop response of the front wheel car model for initial condition: $(x_1, x_2, x_3, x_4) = (0.3, 0.4, 0.2, 0.5)$

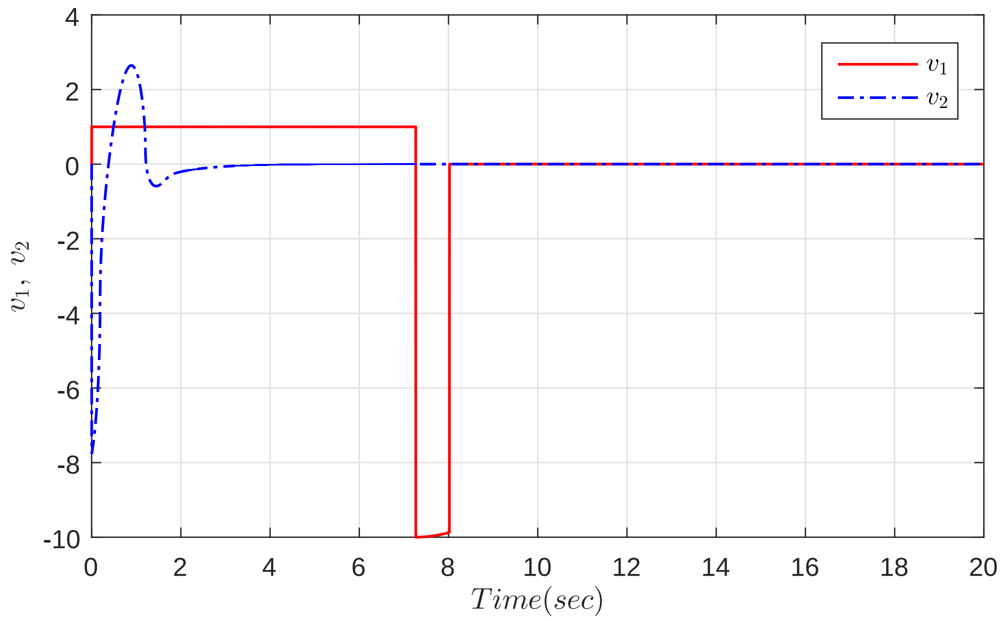

FIGURE 6.8: Control input: v_1, v_2

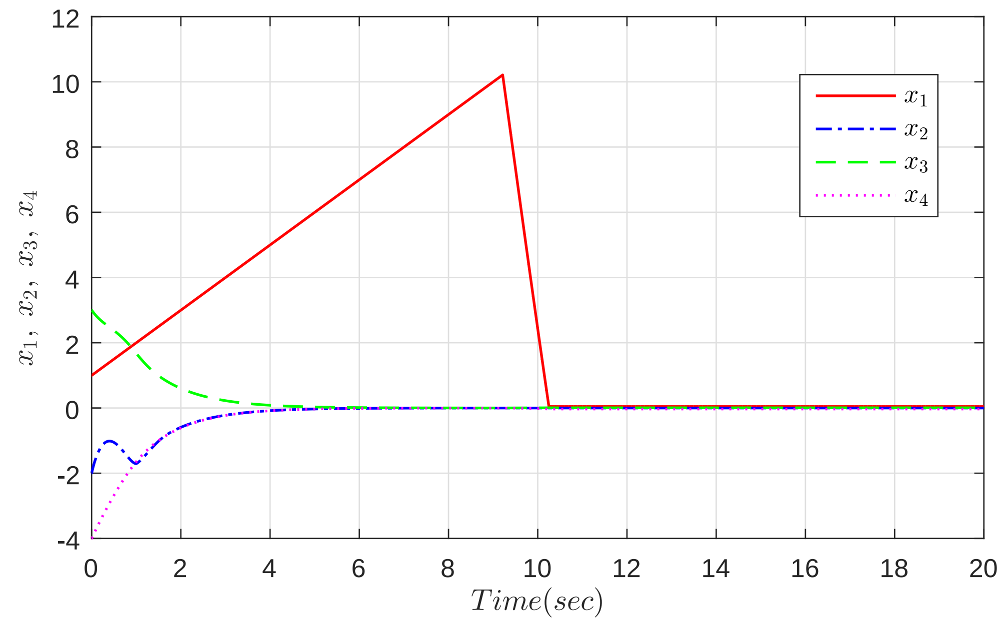

FIGURE 6.9: Closed loop response of the front wheel car model for initial condition: $(x_1, x_2, x_3, x_4) = (1, -2, 3, -4)$

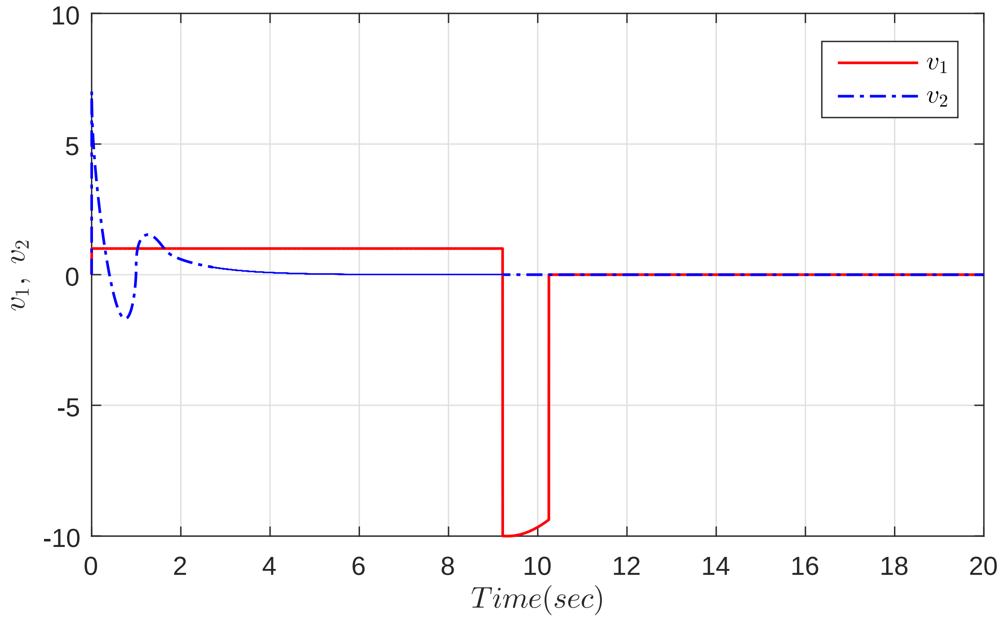

FIGURE 6.10: Control input: v_1, v_2

6.4.3 A Firetruck Model

6.4.3.1 A Kinematics Model of a Firetruck

The firetruck model as shown in Fig. 6.11 represents a six dimensional non-holonomic system. By defining the state variables: $z = (z_1, z_2, z_3, z_4, z_5, z_6)^T = (x, y, \varphi_0, \theta_0, \varphi_1, \theta_1)^T$ and assuming $l_0 = l_1 = 1$ in firetruck model as given in [120], we have the following:

$$\dot{z} = g_0(z)u_1 + g_2(z)u_2 + g_3(z)u_3 \tag{6.34}$$

$$\text{where, } g_0(z) = \begin{bmatrix} \cos z_4 \\ \sin z_4 \\ 0 \\ \tan z_3 \\ 0 \\ -\sin(z_6 - z_4 + z_5)\sec z_5 \end{bmatrix}, \quad g_2(z) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad g_3(z) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

where,

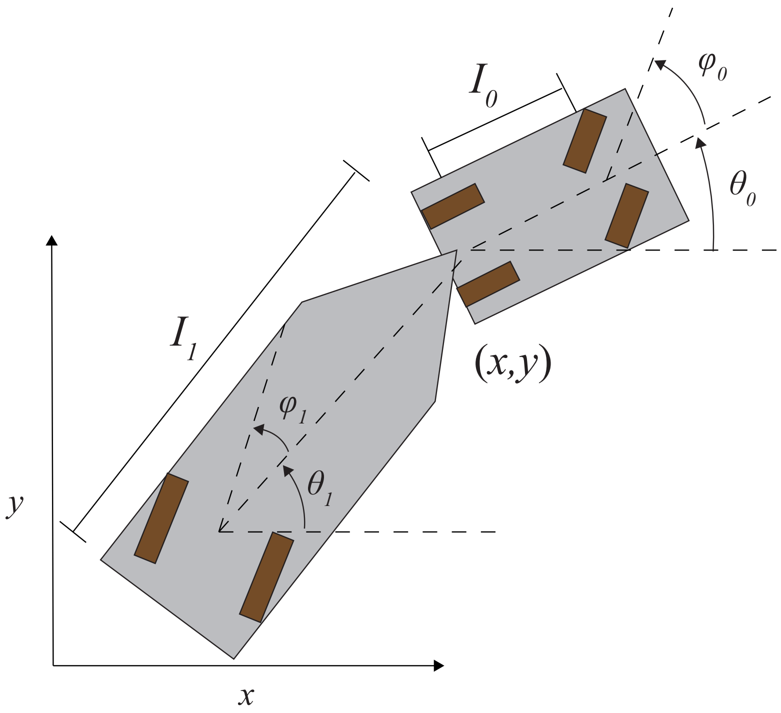

FIGURE 6.11: The schematics of firetruck model

(x, y): the center coordinate of the rear axle of the cab

φ_0: the steering angle of the front wheel of the cab

θ_0: the orientation of the cab body with respect to x-axis

φ_1: the steering angle of the rear wheel of the trailer

θ_1: the orientation of the trailer with respect to x-axis

l_0: the distance between the front and rear axles of the cab

l_1: the distance between the centers of the rear axles of the cab and trailer

The system (6.34) can be rewritten as:

$$\dot{z} = g_1(z)\bar{u}_1 + g_2(z)u_2 + g_3(z)u_3 \tag{6.35}$$

where

$$g_1(z) = \frac{g_0(z)}{\cos z_4} = \begin{bmatrix} 1 \\ \tan z_4 \\ 0 \\ \tan z_3 \sec z_4 \\ 0 \\ -\sin(z_6 - z_4 + z_5) \sec z_5 \sec z_4 \end{bmatrix},$$

$$g_2(z) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad g_3(z) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{u}_1 = \cos z_4 u_1$$

6.4.3.2 Conversion into Chained form

Consider the following transformation:

$$\begin{aligned} x_1 &= z_1 \\ x_2 &= \sec^3 z_4 \tan z_3 \\ x_3 &= \tan z_4 \\ x_4 &= z_2 \\ x_5 &= -\sin(z_5 - z_4 + z_6) \sec z_5 \sec z_4 \\ x_6 &= z_6 \end{aligned} \tag{6.36}$$

$$\begin{aligned} v_1 &= \bar{u}_1 = \cos z_4 u_1 \\ v_2 &= a_1 \bar{u}_1 + a_2 u_2 \\ v_3 &= a_3 \bar{u}_1 + a_4 u_3 \end{aligned}$$

where,

$$
\begin{aligned}
a_1 &= 3\tan^2 z_3 \tan z_4 \sec^4 z_4 \\
a_2 &= \sec^2 z_3 \sec^3 z_4 \\
a_3 &= \cos(z_5 - z_4 + z_6)\tan z_3 \sec z_5 \sec^2 z_4 + \cos(z_5 - z_4 + z_6)\sin(z_5 - z_4 + z_6)\sec^2 z_5 \sec^2 z_4 \\
&\quad - \sin(z_5 - z_4 + z_6)\sec z_5 \sec^2 z_4 \tan z_3 \tan z_4 \\
a_4 &= -\cos(z_5 - z_4 + z_6)\sec z_4 \sec z_5 - \sin(z_5 - z_4 + z_6)\sec z_5 \sec z_4 \tan z_5
\end{aligned}
$$

Transform the nonholonomic system (6.36) into the following chained form system:

$$
\begin{aligned}
\dot{x}_1 &= v_1 \\
\dot{x}_2 &= v_2 \\
\dot{x}_3 &= x_2 v_1 \\
\dot{x}_4 &= x_3 v_1 \\
\dot{x}_5 &= v_3 \\
\dot{x}_6 &= x_5 v_1
\end{aligned}
\tag{6.37}
$$

The proposed algorithm was applied with the sliding surface for subsystem S_2 as:

$$
s = x_6 + 4x_5 + 6x_4 + 4x_3 + x_2 \tag{6.38}
$$

6.4.3.3 Simulation Results

Fig. 6.12 and Fig. 6.14 shows the response of system states when proposed algorithm is applied. The initial conditions are chosen as $(x_1, x_2, x_3, x_4, x_5, x_6) = (0.8, 0.5, 0.7, 0.3, 0.6, 0.4)$ and $[x_1(0), x_2(0), x_2(0), x_3(0), x_4(0), x_5(0), x_6(0)] = [1, -5, 2, -3, 3, 2]$. Fig. 6.8 and Fig. 6.10 shows the control inputs v_1, v_2 with gains $k_1 = 4$, $k_2 = 10$. From the figures, we can see that under the constructed controller, the solution process of the closed-loop system converges to zero almost surely.

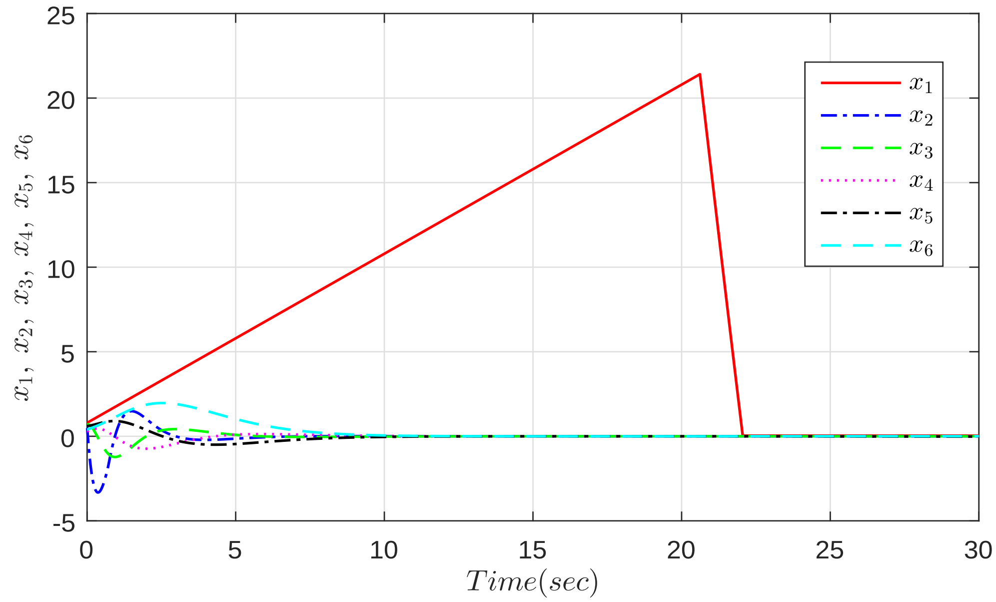

FIGURE 6.12: Closed loop response of the firetruck model for initial condition: $(x_1, x_2, x_3, x_4, x_5, x_6) = (0.8, 0.5, 0.7, 0.3, 0.6, 0.4)$

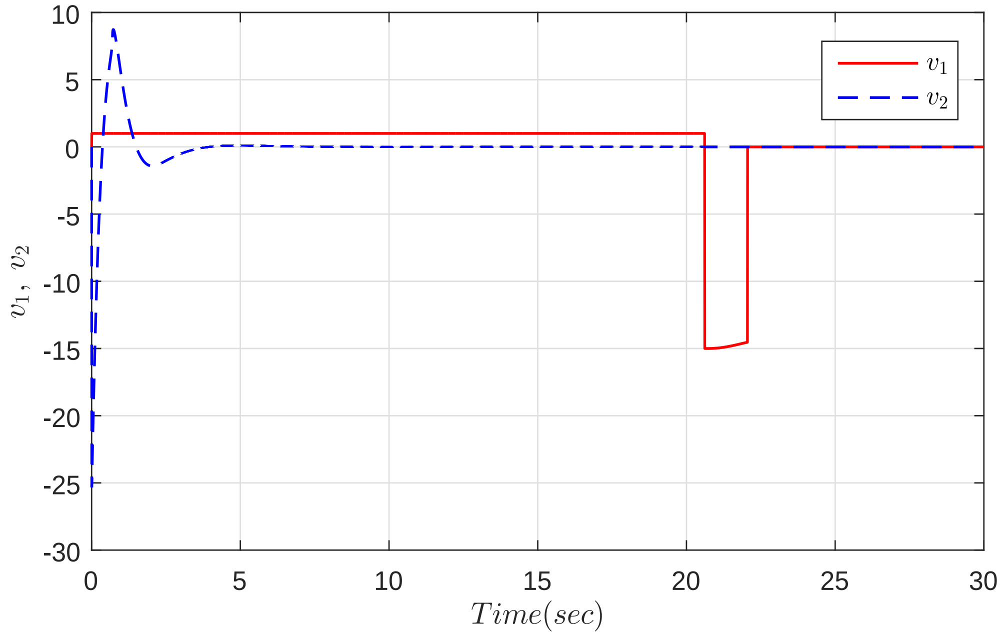

FIGURE 6.13: Control input: v_1, v_2

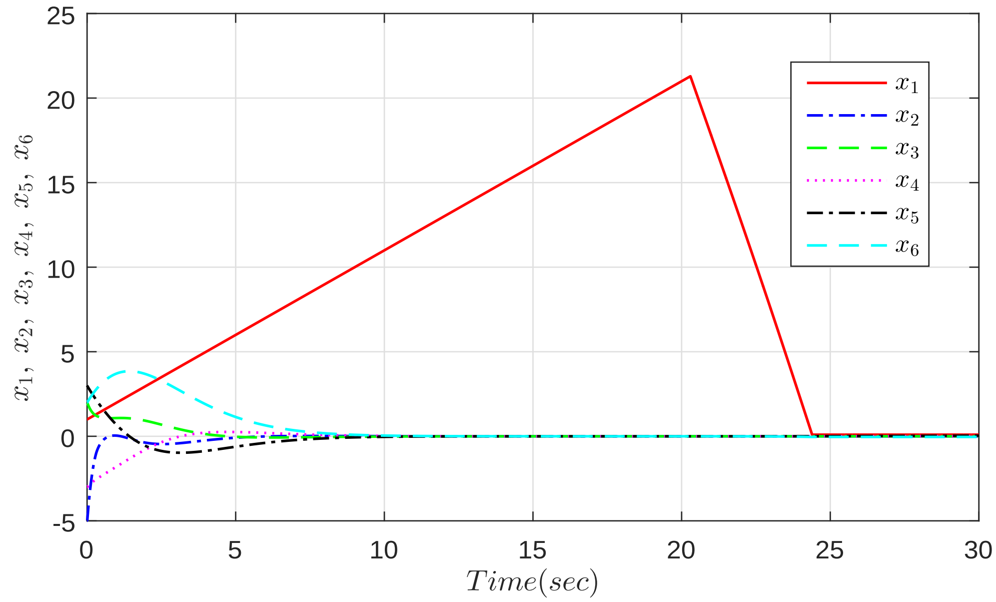

FIGURE 6.14: Closed loop response of the firetruck model for initial condition: $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, -5, 2, -3, 3, 2)$

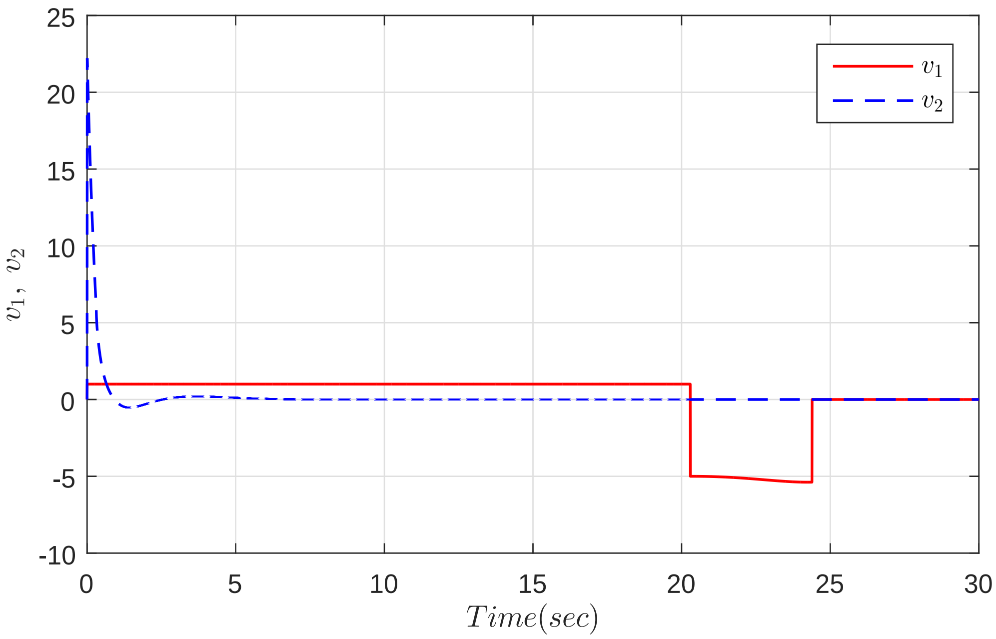

FIGURE 6.15: Control input: v_1, v_2

6.4.4 Comparison between adaptive integral sliding mode control and a smooth super twisting sliding mode control results: A front-wheel car example

Fig. 6.16 and Fig. 6.17 show the results of the front-wheel car model and represent that the states converges with settling time of 7 sec and 8 sec. The control efforts for both the techniques are shown in Fig. 6.18 and Fig. 6.19. The gain of Fig. 6.16 to be set are $k = 5$, $k_1 = 3$ and $k_2 = 3$ while the gains of Fig. 6.17 are adjusted as $k_1 = 4$, $k_2 = 10$. From Fig. 6.16, it has shown that all the states are asymptotically converging to equilibrium point but initially the control effort as shown in Fig. 6.18 chatters a little bit as compared to the control effort shown in Fig. 6.19. In Fig. 6.17, one state variable diverge for a short period of time but then converge to equilibrium point when the control input has been applied.

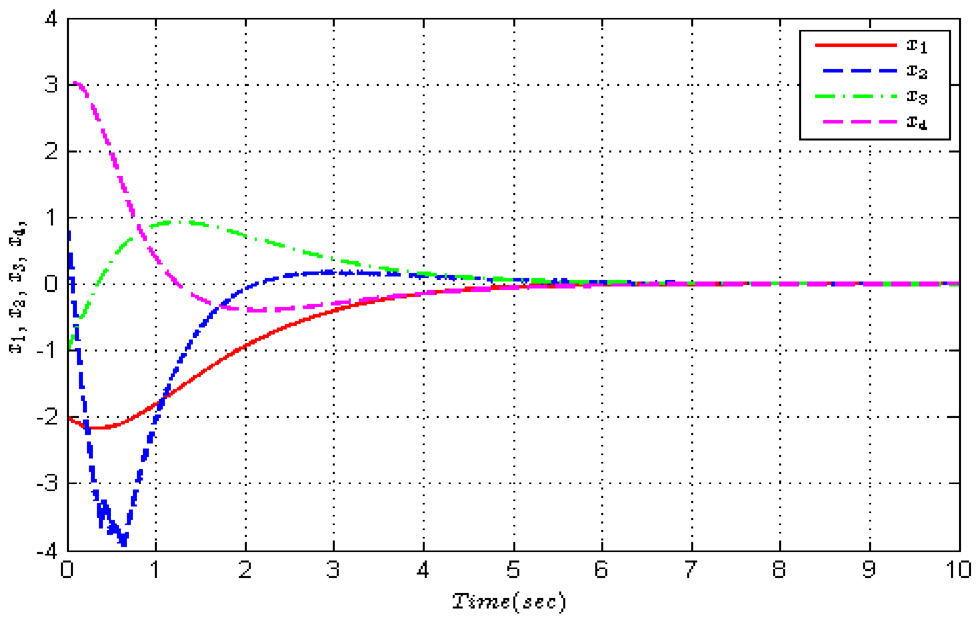

FIGURE 6.16: Closed loop response of the front wheel car model for initial condition: $(x_1, x_2, x_3, x_4) = (0.3, 0.4, 0.2, 0.5)$

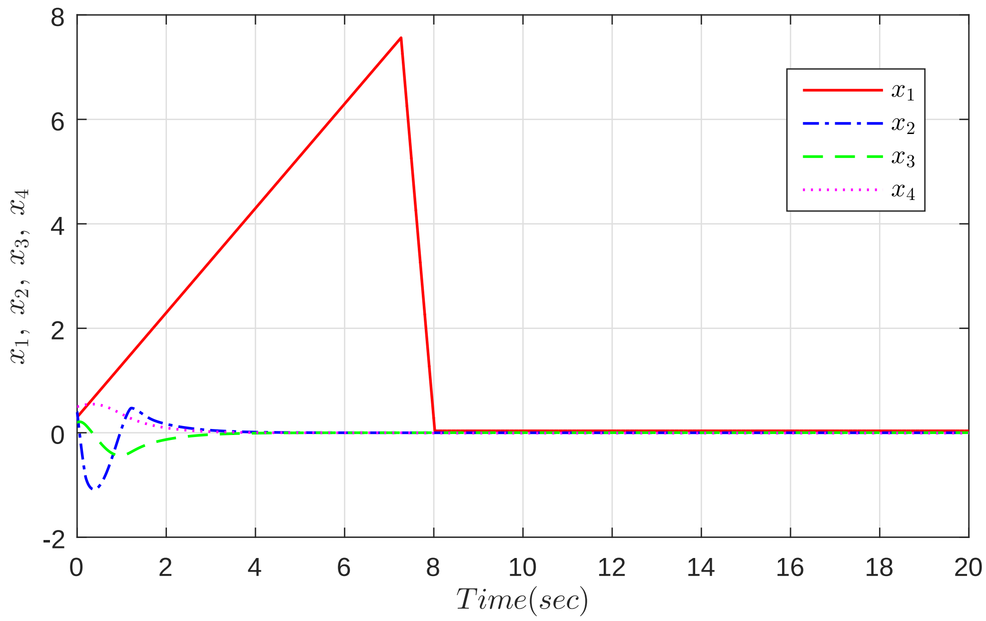

FIGURE 6.17: Closed loop response of the front wheel car model for initial condition: $(x_1, x_2, x_3, x_4) = (0.3, 0.4, 0.2, 0.5)$

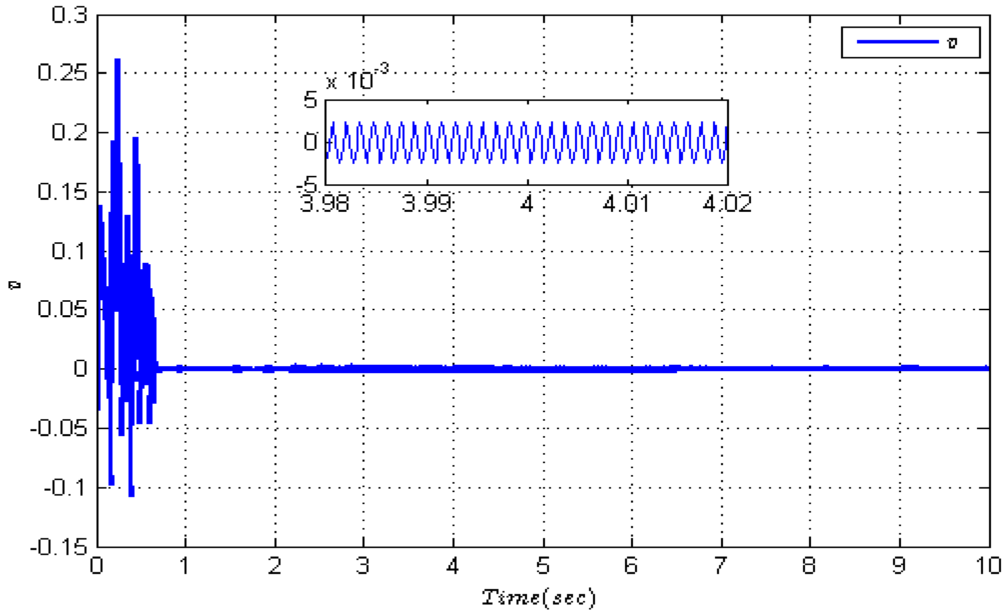

FIGURE 6.18: Control input: v_1, v_2

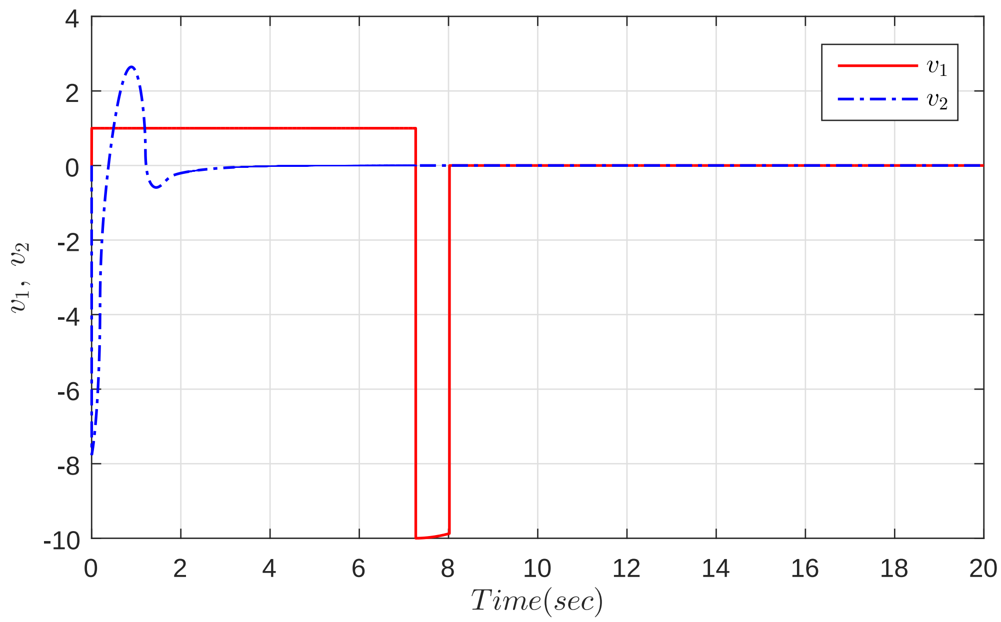

FIGURE 6.19: Control input: v_1, v_2

6.5 Summary

A smooth super twisting sliding mode based stabilizing control for nonholonomic systems which are transformable into chained form is presented. The proposed method is tested on three nonholonomic systems which are transformable into chained form; a two wheel car model, a model of four-wheel car and a fire truck model. The aim is to stabilize the nonholonomic systems to any desired value from any initial state. The simulation results show the correctness and the effectiveness of the proposed controller.

Chapter 7

Conclusion and Future Work

7.1 Conclusion

The goal of this dissertation is to propose new and novel feedback control algorithms for the stabilization of nonholonomic systems. Control and stabilization of these systems is a difficult problem, as these systems are not stabilizable by continuous static state feedback control laws. Furthermore, the addition of drift term makes stabilization of these systems even more challenging. Nevertheless, finding a feedback or coordinate transformation that transform the system into any design friendly form actually facilitates in designing the control law. The transformation of these systems considerably simplifies the kinematics of these systems and also generalizes the control design process such that it can be applied to a whole class of first-order nonholonomic systems.

After reviewing the existing methods in the literature, firstly, a nonholonomic system with drift is considered to gain insight into and create interest to deal with these systems. An adaptive backstepping technique is used to stabilize the nonholonomic systems with drift. In addition to the adaptive backstepping method, sliding mode control technique is also used for the design of feedback control laws for nonholonomic systems. In contrast to the basic sliding mode control, we use integral sliding mode control, and the smooth super twisting sliding mode control approaches for nonholonomic systems. For the stabilization of nonholonomic systems, secondly, an adaptive integral sliding mode based approach is presented. Lastly, these systems are transformed into chained form and then the smooth super twisting sliding mode control approach is used in order to stabilize nonholonomic systems.

The theoretical results of this thesis are applied to challenging control problems of a rigid body, an extended nonholonomic double integrator and for a variety of wheeled mobile robots. Detailed control design and simulation results are provided for each of these systems. Among these examples, the stabilization of unicycle model, front wheel car model, car with trailer model and a firetruck model to the origin is demonstrated. Simulation results prove the effectiveness of the proposed techniques.

7.2 Future Work

Future research will include in-depth study of other interesting nonholonomic systems, such as the snake-board, ball and the plate. More complex systems like multi-finger hands grasping object and space robots are also of high-interest. Stabilization of nonholonomic mechanical systems subject to non-classical constraints, i.e., when the rolling contact occurs through an elastic surface and the problem of global stabilization for very general 3-dimensional systems with two control signals is also a field of immense interest in the control or stabilization of nonholonomic systems.

A list of a few topics for future research is:

1. Analysis of the robustness properties with respect to both model error and external disturbances.

2. Generalization of control strategies for 2^{nd} order or higher order nonholonomic systems.

3. Accommodation of other control objectives such as trajectory tracking and motion planning.

4. Observer-based control when states are not accessible for measurement.

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